Appendix A

Oscillator Equation and Absorption Rate

The Newton equation of motion for a particle of mass m and charge e, acted upon by an elastic restoring force $-m\omega_o^2 z$ and an external electric field $E_z(t)$, is

$$\ddot{z} + \omega_o^2 z = \frac{e}{m}E_z(t) + \frac{e}{m}E_{\mathrm{RR}}(t). \tag{A.1}$$

For simplicity, and to follow Planck, Einstein, and Hopf, we assume the particle is constrained to one-dimensional motion.

The field $E_{\mathrm{RR}}(t)$ in (A.1) is the field of radiation reaction, i.e., the electric field produced by the charged particle at the position of the particle. In other words, it is the electric field that the charge exerts on itself. For our purposes here a simplified derivation and expression for this field will suffice. A more detailed derivation is given in Appendix D.

We recall first the expression (1.8) for the rate at which an accelerating charge radiates electromagnetic energy. The energy radiated in the time interval from t_1 to t_2 is

$$W_{\mathrm{EM}}(t_2, t_1) = \frac{2e^2}{3c^3}\int_{t_1}^{t_2} \ddot{z}(t)^2 dt = \frac{2e^2}{3c^3}[\ddot{z}(t)\dot{z}(t)\,|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dddot{z}\,(t)\dot{z}(t)dt], \tag{A.2}$$

where the second equality follows from an integration by parts. We assume the motion of the charge is periodic and choose $t_2 - t_1$ to be an integral number of periods, in which case

$$W_{\mathrm{EM}}(t_2, t_1) = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2} \dddot{z}\,(t)\dot{z}(t)dt. \tag{A.3}$$

The change in energy of the charge, $-W_{\rm EM}$, is attributed to the force $eE_{\rm RR}(t)$ of radiation reaction:

$$-W_{\rm EM}(t_2,t_1)=\frac{2e^2}{3c^3}\int_{t_1}^{t_2}\dddot{z}(t)\dot{z}(t)dt=\int_{t_1}^{t_2}eE_{\rm RR}(t)\dot{z}(t)dt \tag{A.4}$$

or

$$E_{\rm RR}(t)=\frac{2e}{3c^3}\dddot{z}(t). \tag{A.5}$$

Although this expression for the radiation reaction field was derived under the assumption of periodic motion, it actually holds more generally, as discussed in Chapter 5 and Appendix D. When it is used in (A.1), we obtain the equation (1.41) used by Planck, Einstein and Stern, and others.

For the case of a monochromatic applied field $E_z(t)=E_{z\omega}\cos(\omega t+\theta_\omega)$, equation (1.41) has the solution

$$z(t)=-\frac{e}{m}\text{Re}\left[\frac{E_{z\omega}e^{-i(\omega t+\theta_\omega)}}{\omega^2-\omega_o^2+i\gamma\omega^3}\right], \tag{A.6}$$

so that the rate (force times velocity) at which the oscillator absorbs energy from the field is found after some simple algebra to be

$$\dot{W}_{\rm A}=e\dot{z}(t)E_z(t)\to\frac{e^2}{2m}\frac{\gamma\omega^4E_{z\omega}^2}{(\omega^2-\omega_o^2)^2+\gamma^2\omega^6}, \tag{A.7}$$

where we have taken an average over the oscillations of the field, replacing $\cos^2(\omega t+\theta_\omega)$ by $1/2$ and $\sin(\omega t+\theta_\omega)\cos(\omega t+\theta_\omega)$ by 0.

Now suppose the applied field has a broad distribution of frequencies, with energy density in the interval $[\omega,\omega+d\omega]$ given by $\rho(\omega)d\omega=E_{z\omega}^2/8\pi$. In this case (A.7) is replaced by

$$\dot{W}_{\rm A}=\frac{4\pi e^2}{m}\gamma\int_0^\infty\frac{\omega^4\rho(\omega)d\omega}{(\omega^2-\omega_o^2)^2+\gamma^2\omega^6}. \tag{A.8}$$

The time $\gamma=2e^2/3mc^3=6.3\times10^{-24}$ sec is so short that, for natural oscillation frequencies ω_o of interest, $\gamma\omega_o<<1$. Furthermore $\rho(\omega)$ may be assumed to be flat compared with the sharply peaked function

$$\frac{\omega^4}{(\omega^2-\omega_o^2)^2+\gamma^2\omega^6}\cong\frac{\omega_o^4}{4\omega_o^2(\omega-\omega_o)^2+\gamma^2\omega_o^6} \tag{A.9}$$

in the integrand of (A.8), so that

$$\begin{aligned}\dot{W}_{\rm A}&\cong\frac{\pi e^2\gamma}{m}\omega_o^2\rho(\omega_o)\int_0^\infty\frac{d\omega}{(\omega-\omega_o)^2+\gamma^2\omega_o^4/4}\cong\frac{\pi e^2\gamma}{m}\omega_o^2\rho(\omega_o)\left(\frac{2\pi}{\gamma\omega_o^2}\right)\\&=\frac{2\pi^2e^2}{m}\rho(\omega_o)=\frac{\pi e^2}{m}\rho(\nu_o)\to\frac{\pi e^2}{3m}\rho(\nu_0).\end{aligned} \tag{A.10}$$

In the last step we have replaced $\rho(\nu_o)$ by $\rho(\nu_o)/3$, where now the spectral energy density is defined by $\rho(\omega)d\omega = (E_{x\omega}^2 + E_{y\omega}^2 + E_{z\omega}^2)/8\pi = 3E_{z\omega}^2/8\pi$ for (isotropic and unpolarized) thermal radiation. We have thus arrived at equation (1.7) for the energy absorption rate.

By replacing e^2/m by $e^2 f/m$ in equation (1.7), where f is the oscillator strength of an atomic transition of frequency ω_o, we obtain the energy absorption rate given by quantum mechanics up to second order in perturbation theory.[1]

[1] See, for instance, M. Cray, M.-L. Shih, and P. W. Milonni, *Am. J. Phys.* **50**, 1016 (1982).