4

The Lagrangian formalism

We have seen that in the Newtonian approach one has to take into account of two kinds of forces: the given force (denoted by $\mathbf{F}$) and the constraint force (denoted by $\mathbf{F}^c$). The given force includes the externally impressed forces, and the forces of interaction between mass points through springs and frictional elements. The Newtonian equation for each mass point is

$$m_j\ddot{\mathbf{r}}_j - \mathbf{F}_j = \mathbf{F}_j^c \tag{4.1}$$

where $\mathbf{F}_j$ and $\mathbf{F}_j^c$ are the given force and the constraint force respectively, on the jth mass point. If there are N number of mass points there will be N such equations. Summing up the N equations, we get

$$\sum_{j=1}^{N}(m_j\ddot{\mathbf{r}}_j - \mathbf{F}_j) = \sum_{j=1}^{N}\mathbf{F}_j^c. \tag{4.2}$$

This is the basic Newtonian equation for a system comprising of N mass points.

About a century after Newton, Lagrange showed that it is much more convenient to formulate differential equations in terms of the two basic forms of energies contained in a system: the kinetic energy and the potential energy. While there may be a large number of interacting forces in a system, there are only two forms of energy. Thus if the central concept of Newtonian mechanics can be expressed in terms of the energies, the formulation would be much simpler. In the following sections, we will explore this conceptual breakthrough to develop a powerful method of formulating differential equations of physical systems.

4.1 Elements of the Lagrangian Approach

4.1.1 Motivation

It is clear from Chapter 2 that if one tries to write the dynamical equations of any system in the Newtonian way, one faces a few practical difficulties :

1. All the constraint forces have to be included in the formulation. This would be difficult, for in most cases the constraint forces are not easily quantifiable. Moreover, in a

complicated system there may be a large number of constraint forces. Accounting for all of them is a cumbersome exercise.

2. In a system with interconnected elements, the components interact with each other through forces which are to be categorized under "given forces" $\mathbf{F}_j$. These include the force of interaction of between masses through springs or frictional elements. We need to take into account all these forces, and the method becomes unmanageable with the increase in the number of interacting elements.

   It would naturally be more convenient to express the forces in terms of some quantity of lesser complexity. The energies in a system, i.e., the kinetic energy and the potential energy, are *scalar* quantities which are easily expressible in terms of the system coordinates. It would therefore be convenient to express the dynamical equations in terms of the energies of the system.

3. For a N-body problem one would have to obtain $3N$ differential equations though we have seen that the existence of holonomic constraints reduce the number of configuration coordinates. It is desirable to utilize the advantage offered by the holonomic systems, i.e., the number of system equations should be reduced as far as possible. This can be achieved by expressing the system equations in terms of a new coordinate system defined on the constraint surface.

The methodology for overcoming these problems was developed through the work of post-Newtonian scientists like Bernoulli, D'Alembert, Lagrange, Hamilton and others. This will be taken up in the next section.

4.1.2 The concept of admissible motions

We take up the first problem, that is, elimination of the constraint forces from the system model. We notice that in most cases the constraint force does no work because it is perpendicular to the directions of motion allowed by the constraint. Let us illustrate it with three examples.

Example 4.1 *(a) A pendulum's constraint allows motions that are always orthogonal to the tension in the string. Hence the work done by the constraint force is zero.*

(b) In a pulley, the constraint forces act on two bodies along the string. Since the direction of the constraint force and the possible displacement are along the same line, the work done by the constraint force exists for both the masses individually. However, since they are equal and opposite, the sum is always zero.

(c) For a body sliding along a frictionless surface, the force of reaction is always perpendicular to the direction of motion. Hence the work due to the constraint force vanishes. If there is friction at the surface, the force of reaction would not be perpendicular to the motion. In that case we'll consider the perpendicular component of the reaction force as the force due to constraint and include the force due to friction (acting tangentially to the surface) in the category of given force $\mathbf{F}_j$.

Notice that in the Example 4.1(a), if the point of suspension of the pendulum moves with time, the constraint force will no longer be orthogonal to possible directions of motion.

Same is the case if the surface in Example 4.1(c) moves with time. Still we would like to obtain a general framework in which the constraint force can be eliminated. We do this by considering the *geometrically admissible displacements* compatible with the constraints, for which the work due to constraint forces would be identically zero.

For example, an geometrically admissible displacement for a spherical pendulum can be any displacement tangential to the spherical constraint surface. But if it is a pendulum with an oscillating support, then the admissible displacement will be in relation to a given position of the pendulum. If we take a still photograph of any position of the pendulum, then any admissible displacement from that position will have to be perpendicular to the chord along which the constraint force acts. Clearly, the work done by the constraint force along an admissible displacement would always be zero.

A geometrically admissible quantity is denoted by the symbol δ. Thus an admissible displacement is written as $\delta\mathbf{r}$. The work done by the constraint force at an admissible displacement is given by $\mathbf{F}^c\delta\mathbf{r}$.

Next, we convert (4.2) into an equation involving work by multiplying by the admissible displacement

$$\sum_{j=1}^{N}(m_j\ddot{\mathbf{r}}_j - \mathbf{F}_j)\cdot\delta\mathbf{r}_j = \sum_{j=1}^{N}\mathbf{F}_j^c\cdot\delta\mathbf{r}_j \tag{4.3}$$

The right hand side of this equation would be zero, eliminating the constraint forces from our formulation. We thus obtain the equation

$$\sum_{j=1}^{N}(m_j\ddot{\mathbf{r}}_j - \mathbf{F}_j).\delta\mathbf{r}_j = 0 \tag{4.4}$$

4.1.3 The generalized coordinates

We now turn to the problem of minimizing the number of coordinates. In (4.2), each $\mathbf{r}_j$ had three components making the dimension of the configuration space $3N$. We have earlier seen that each holonomic constraint defines a surface in this $3N$-dimensional space. By taking the intersection of all such constraint surfaces we get a space of dimension $3N - h$ (h being the number of holonomic constraints) on which the configuration point must lie. We define a new coordinate system on this $(3N - h)$ dimensional space in which q_i are the independent coordinates. These are called *generalized coordinates*. This procedure has the effect of flattening out the constraint surface into a new $3N - h = n$ dimensional space whose axes are the generalized coordinates (Fig.4.1).

Thus the minimum number of independent coordinates in any system, consistent with the constraints, give the generalized coordinates. To be exact, *any set of coordinates* $\{q_1, q_2, q_3 ... q_n\}$ *is called a set of generalized coordinates of a system if and only if the number* n *of its members is necessary and sufficient to define the configuration or positional status of the system uniquely*.

There is no unique choice of the generalized coordinates, since any convenient set of variables that can specify the positional configuration of a system can serve as the generalized coordinates. These need not be the cartesian coordinates of the mass points. In some cases of mechanical systems it may be convenient to express the system configuration in terms of polar coordinates r and θ. In systems consisting of two or more inertial elements, the position

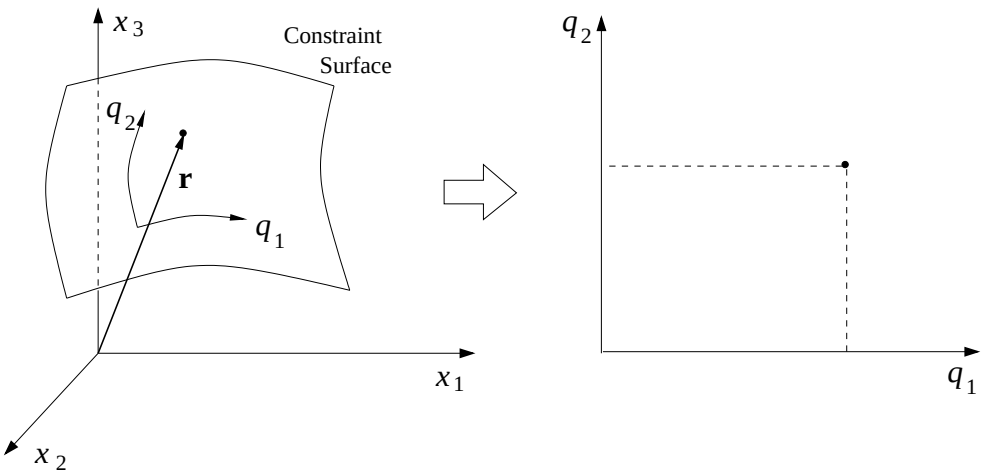


Figure 4.1 Generalized coordinates defined on the surface of constraint.

of some of the masses may be expressed in rectangular coordinates and some others in polar coordinates. We'll see later that in electrical circuits, the charges flowing in loops are the natural choice for the generalized position coordinates.

In such representation, the generalized velocities are merely the time derivatives of the chosen position coordinates. Thus if a mass is represented by polar coordinates r and θ, then the generalized velocities are just $\dot{r}$ and $\dot{\theta}$, and not the actual velocities in the radial and circumferencial directions (the velocity in the circumferencial direction is $r\dot{\theta}$).

The transformation of the system representation from $3N$ coordinates to $3N - h = n$ generalized coordinates can be achieved by expressing the old vectors x_k in terms of the new coordinates q_i in the form

$$x_k = x_k(q_1, q_2, q_3 \cdots q_n, t),$$

where k takes values between 1 and $3N$.

It will however be convenient to put together the coordinates of each mass-point to form vectors $\mathbf{r}_j$, which can then be expressed as functions of the new coordinates:

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, q_3 ... q_n, t)$$

$$\vdots$$

$$\mathbf{r}_N = \mathbf{r}_N(q_1, q_2, q_3 ... q_n, t)$$

or in short

$$\mathbf{r}_j = \mathbf{r}_j(q_i, t)$$

where j varies from 1 to N and i varies from 1 to n.

By differentiation with respect to t, we get the velocities of the mass points in terms of the generalized coordinates as

$$\dot{\mathbf{r}}_j = \frac{d\mathbf{r}_j}{dt} = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_j}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}_j}{\partial t}$$

Note that the last term exists only in rheonomic systems where the constraint equations have expicit dependence on time. This term is independent of $\dot{q}_i$, and $\dot{q}_k$ is also independent of $\dot{q}_i$. Partially differentiating this equation with respect to the generalized velocities $\dot{q}_i$ we therefore get

$$\frac{\partial \dot{\mathbf{r}_j}}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_j}{\partial q_i} \tag{4.5}$$

We'll need these equations for future reference.

In the generalized coordinate system, the generalized force along the coordinates can be expressed in terms of the given forces on the mass points ($\mathbf{F}_j$) as

$$Q_i = \sum_{j=1}^{N} \mathbf{F}_j \frac{\partial \mathbf{r}_j}{\partial q_i} \tag{4.6}$$

Likewise, an admissible displacement $\delta \mathbf{r}_j$ in the old set of coordinates would be related to admissible displacements δq_i in the new generalized coordinates by

$$\delta \mathbf{r}_j = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_j}{\partial q_i} \delta q_i \tag{4.7}$$

This equation does not contain any time variation term δt because by definition a admissible displacement involves changes in space coordinates only.

This completes our conversion of the larger $3N$-dimensional coordinate system to the smaller n-dimensional coordinate system. We will use these relationships later to formulate the dynamical equations in the generalized coordinate system.

4.1.4 Dynamical equations in terms of energies

Now we take up the third objective and start from equation (4.4) derived earlier. Let us reproduce it here for the sake of convenience:

$$\sum_{j=1}^{N} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j).\delta \mathbf{r}_j = 0.$$

Substituting the expression for admissible displacement from (4.7) we get (4.4) in terms of generalized coordinates as

$$\sum_{j=1}^{N} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j) \sum_{i=1}^{n} \frac{\partial \mathbf{r}_j}{\partial q_i} \delta q_i = 0 \tag{4.8}$$

The summation over i can be brought forward, giving

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{N} (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j) \frac{\partial \mathbf{r}_j}{\partial q_i} \right] \delta q_i = 0 \tag{4.9}$$

or,

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{N}(m_j\ddot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}-\mathbf{F}_j\frac{\partial\mathbf{r}_j}{\partial q_i})\right]\delta q_i=0 \tag{4.10}$$

Now consider the time derivative

$$\frac{d}{dt}\left(\dot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}\right)=\ddot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}+\dot{\mathbf{r}}_j\frac{\partial\dot{\mathbf{r}}_j}{\partial q_i}$$

or,

$$\ddot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}=\frac{d}{dt}\left(\dot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}\right)-\dot{\mathbf{r}}_j\frac{\partial\dot{\mathbf{r}}_j}{\partial q_i}$$

Substituting into (4.10) we get

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{N}m_j\frac{d}{dt}\left(\dot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}\right)-\sum_{j=1}^{N}m_j\dot{\mathbf{r}}_j\frac{\partial\dot{\mathbf{r}}_j}{\partial q_i}-\sum_{j=1}^{N}\mathbf{F}_j\frac{\partial\mathbf{r}_j}{\partial q_i}\right]\delta q_i=0 \tag{4.11}$$

Now let us analyse each term in (4.11). We observe that the kinetic energy T is given by

$$T=\sum_{j=1}^{N}\frac{1}{2}m_j\dot{\mathbf{r}}_j^2$$

Differentiating with respect to the generalized velocities we get

$$\frac{\partial T}{\partial\dot{q}_i}=\sum_{j=1}^{N}m_j\dot{\mathbf{r}}_j\frac{\partial\dot{\mathbf{r}}_j}{\partial\dot{q}_i}$$

By (4.5) we get

$$\frac{\partial T}{\partial\dot{q}_i}=\sum_{j=1}^{N}m_j\dot{\mathbf{r}}_j\frac{\partial\mathbf{r}_j}{\partial q_i}$$

Again, differentiating the expression for kinetic energy with respect to the generalized coordinates we get

$$\frac{\partial T}{\partial q_i}=\sum_{j=1}^{N}m_j\dot{\mathbf{r}}_j\frac{\partial\dot{\mathbf{r}}_j}{\partial q_i}$$

Lastly, by (4.6) the term $\sum_{j=1}^{N}\mathbf{F}_j\frac{\partial\mathbf{r}_j}{\partial q_i}$ turns out to be nothing but the force along the jth generalized coordinate. This is called the generalized force Q_i.
Substituting these for the three terms in (4.11) we have

$$\sum_{i=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}-Q_i\right\}\delta q_i=0$$

Since the generalized coordinates are independent, any admissible displacement along the ith coordinate (δq_i) would be independent of admissible displacements along other generalized coordinates. Therefore the only way the summation in the above equation can vanish is if the terms in the curly bracket are zero. Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i = 0 \tag{4.12}$$

This is the Lagrangian equation in its most general form. Here the generalized force Q_i contains all the given forces in the system acting along the i-th coordinate.

The given forces may be of three categories:

1. Forces of interaction between different mass-points of the system. These happen generally by means of springs or elements of spring-like characteristics.
2. External forces, including force due to gravity.
3. Forces due to friction.

Forces of the first two categories can be derived from a scalar "potential function" denoted as V. The force due to gravity can be obtained from the gravitational potential; the force exerted by a spring can be obtained from the potential energy stored in the spring. We'll show in section 4.5 that externally impressed forces (incluing electromotive forces) can also be obtained from a suitably defined potential function. The generalized forces are derived from the potential as

$$Q_i = -\frac{\partial V}{\partial q_i}$$

Systems where the generalized forces are obtainable from a scalar potential are called *conservative systems*.

For conservative systems (i.e., where forces are only of the first and second categories) (4.12) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{4.13}$$

To make this equation further compact, we define a new function called the *Lagrangian* function[1] as

$$\mathcal{L} = T - V$$

Since V is independent of the generalized velocities $\dot{q}_i$,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \tag{4.14}$$

Hence (4.13) becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4.15}$$

This is the Lagrangian equation for conservative systems.

[1] We denote the Lagrangian by script $\mathcal{L}$ to distinguish it from the symbol of inductance, L.

4.2 Obtaining dynamical equations by Lagrangian method

As shown earlier, in the Lagrangian method one does not have to worry about the constraint forces or the forces of interaction between elements of the system. Nor does one have to worry about the change of coordinates. The equations can be derived following a few simple steps given below.

1. Identify a minimum set of generalized coordinates consistent with the constraints.
2. Express the kinetic and potential energies in terms of the generalized coordinates and the generalized velocities, and obtain the Lagrangian function from these.
3. Partially differentiate $\mathcal{L}$ with respect to q_i and $\dot{q}_i$.
4. Write the Lagrangian equation (4.15) for each generalized coordinate.

Let us illustrate these steps with a few examples.

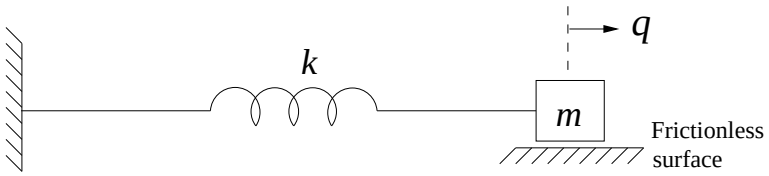


Figure 4.2 The one-dimensional mass-spring system.

Example 4.2 *We shall start with the simple example of a mass-spring system as shown in Fig.4.2. The mass is constrained to move in only one direction and hence the position of the whole system can be defined by one generalized coordinate q, consistent with the constraint. We take $q = 0$ at the position of equilibrium of the mass.*

The motion of the mass is governed by Newton's law: mass×acceleration = force, or

$$m\frac{d^2q}{dt^2} = -F \tag{4.16}$$

Here F, the restoring force exerted by the spring, is negative because it acts in a direction opposite to q. This is the system equation, derived in the Newtonian way.

Now we obtain the equation in the Lagrangian way. When the spring is stretched a distance q, the potential energy stored in the spring is

$$V = \frac{1}{2}kq^2 \tag{4.17}$$

where k is the spring constant.
The kinetic energy is given by

$$T = \frac{1}{2}m\dot{q}^2 \tag{4.18}$$

Hence the Lagrangian function is

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

Now we diferentiate $\mathcal{L}$ by $\dot{q}$ and q to get respectively

$$\frac{d\mathcal{L}}{d\dot{q}} = m\dot{q}$$

$$\frac{d\mathcal{L}}{dq} = -kq = -F$$

Hence the Lagrangian equation (4.15) gives

$$m\ddot{q} + F = 0$$

Which is the same as the Newtonian equation (4.16).

In this example no particular advantage of the Lagrangian methodology is visible. To illustrate the advantage, let's take the following examples.

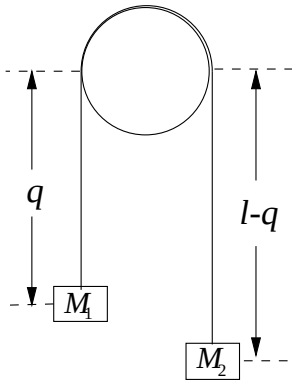


Figure 4.3 Dynamics of the pulley.

Example 4.3 *We consider the motion of a pulley as shown in Fig.4.3. Here if you try to formulate the dynamical equation by the conventional method, you will have to worry about the tension in the string and all that. In contrast, by the above method we would simply note that though there are two masses, the configuration can be exactly defined with only one generalized coordinate q. The position of the other mass is $(l-q)$ where l is the length of the string. We would then write V, T and $\mathcal{L}$ in terms of q and $\dot{q}$ as*

$$V = -M_1gq - M_2g(l-q)$$

$$T = \frac{1}{2}(M_1 + M_2)\dot{q}^2$$

$$\mathcal{L} = \frac{1}{2}(M_1 + M_2)\dot{q}^2 + M_1 g q + M_2 g(l - q)$$

The derivatives with respect to q and $\dot{q}$ are

$$\frac{d\mathcal{L}}{dq} = (M_1 - M_2)g$$

$$\frac{d\mathcal{L}}{d\dot{q}} = (M_1 + M_2)\dot{q}$$

Thus the Lagrangian equation gives

$$(M_1 + M_2)\ddot{q} - (M_1 - M_2)g = 0$$

This is the dynamical equation for the system in Fig.4.3.

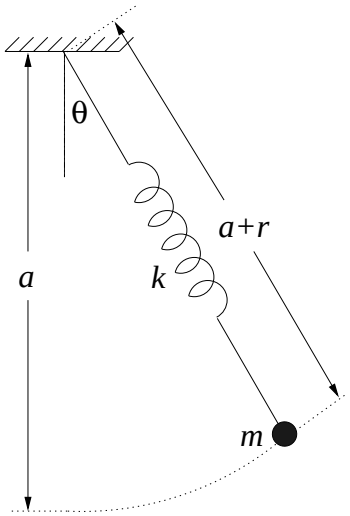


Figure 4.4 System pertaining to the example.

Example 4.4 *For the spring-pendulum system shown in Figure 4.4, we need two configuration coordinates to specify the positional status of the bob. One coordinate is θ and the other coordinate is dependent on the radial distance from the suspension to the bob. There are many possible ways of defining this coordinate, and we choose one of them as follows.*

The spring is at equilibrium in the upright position when $\theta = 0$ and elongation $= mg/k$. Let the total length of the spring in the equilibrium condition be a. Let one of the position coordinates (r) be defined as the deviation from the length a. We take the equilibrium position of the bob as the "datum" level for measuring the potential energy.

In terms of these position coordinates, the radial component of the kinetic energy is $\frac{1}{2}m\dot{r}^2$ and the tangential component is $\frac{1}{2}m[(a + r)\dot{\theta}]^2$. Hence the total kinetic energy is given by

$$T = \frac{1}{2}m[\dot{r}^2 + (a + r)^2\dot{\theta}^2]$$

Note that T is dependent on the velocities $\dot{r}$, $\dot{\theta}$ as well as the position r.

The potential energy has two components — that due to the spring and that due to the gravitational potential. The elongation of the spring from unstretched condition is $(r + mg/k)$. Hence the potential energy stored in the spring is $\frac{1}{2}k(r + mg/k)^2$. When the mass is in equilibrium vertically below the suspension, the gravitational potential is zero. At other positions the height of the mass from the zero-level is $[a-(a+r)\cos\theta]$. Hence the gravitational potential is $mg[a-(a+r)\cos\theta]$. Adding the two we get the total potential as

$$V = \frac{1}{2}k(r + mg/k)^2 - mg(a + r)\cos\theta + mga$$

This gives

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(a + r)^2\dot{\theta}^2 - \frac{1}{2}k(r + mg/k)^2 + mg(a + r)\cos\theta - mga$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{r}} &= m\dot{r} \\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= m(a + r)^2\dot{\theta} \\
\frac{\partial \mathcal{L}}{\partial r} &= m\dot{\theta}^2(a + r) - k(r + mg/k) + mg\cos\theta \\
\frac{\partial \mathcal{L}}{\partial \theta} &= -mg(a + r)\sin\theta
\end{aligned}$$

Hence the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$$

gives

$$m\ddot{r} - m\dot{\theta}^2(a + r) + k(r + mg/k) - mg\cos\theta = 0$$

and the other equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

gives

$$m\frac{d}{dt}[(a + r)^2\dot{\theta}] + mg(a + r)\sin\theta = 0$$

Upon simplification we get the second equation as

$$m(a + r)\ddot{\theta} + 2m\dot{\theta}\dot{r} + mg(a + r)\sin\theta = 0$$

Example 4.5 *We take the system in Fig.4.5. The configuration of the system can be completely defined by the positions of the two masses in the horizontal direction, given by q_1 and q_2. The differential equations governing the dynamics of the system can be expressed in terms of these two variables.*

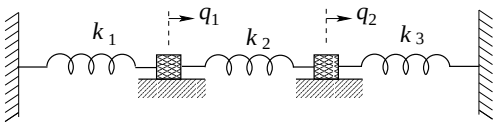


Figure 4.5 A mechanical system with two degrees of freedom.

In this particular case,

$$T = \frac{1}{2}m_1\dot{q_1}^2 + \frac{1}{2}m_2\dot{q_2}^2 \tag{4.19}$$

$$V = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2(q_1 - q_2)^2 + \frac{1}{2}k_3q_2^2 \tag{4.20}$$

We form the partial derivatives

$$\frac{\partial \mathcal{L}}{\partial \dot{q_1}} = m_1\dot{q_1}$$

This is the momentum of the mass m_1.

$$\frac{\partial \mathcal{L}}{\partial q_1} = -k_1q_1 - k_2(q_1 - q_2)$$

This is the restoring force on the mass m_1. *So the Lagrangian equation for the* q_1 *coordinate is*

$$m_1\ddot{q_1} + k_1q_1 + k_2(q_1 - q_2) = 0$$

Similarly, for the q_2 *axis we get*

$$m_2\ddot{q_2} + k_3q_2 - k_2(q_1 - q_2) = 0$$

These are the two equations governing the system dynamics.

Example 4.6 *Derive the dynamical equations of the spring-connected triple pendulum system shown in Fig.4.6.*
Solution: *The minimum set of coordinates that uniquely define the positional status of this system are the three angles* θ_1, θ_2 *and* θ_3. *In terms of these generalized coordinates, the kinetic energy of the system in Fig.4.6 is*

$$T = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$$

The potential energy consists of two parts: the energy due to the gravitational force and the strain energy of the springs. The energy due to gravity is

$$V_g = mgl(1 - \cos\theta_1) + mgl(1 - \cos\theta_2) + mgl(1 - \cos\theta_3)$$

If the angles are small, this can be approximated to

$$V_g \approx \frac{1}{2}mgl(\theta_1^2 + \theta_2^2 + \theta_3^2)$$

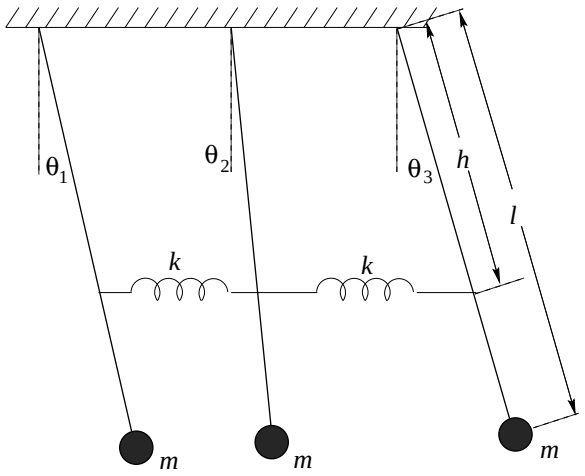


Figure 4.6 System pertaining to the example.

The elongations of the springs are given by

$$h(\sin\theta_2 - \sin\theta_1) \approx h(\theta_2 - \theta_1) \qquad \textit{and}$$

$$h(\sin\theta_3 - \sin\theta_2) \approx h(\theta_3 - \theta_2) \qquad \textit{respectively.}$$

Hence the energy stored in the springs is

$$V_s = \frac{1}{2}kh^2[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2]$$

This gives the total potential energy as

$$V = \frac{1}{2}mgl(\theta_1^2 + \theta_2^2 + \theta_3^2) + \frac{1}{2}kh^2[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2]$$

There are three generalized coordinates and hence the system will be described by three Lagrangian equations. For θ_1, θ_2 and θ_3, the Lagrangian equation gives respectively

$$l^2m\ddot{\theta}_1 + mgl\theta_1 + kh^2(\theta_1 - \theta_2) = 0$$

$$l^2m\ddot{\theta}_2 + mgl\theta_2 + kh^2(\theta_2 - \theta_1) + kh^2(\theta_2 - \theta_3) = 0$$

$$l^2m\ddot{\theta}_3 + mgl\theta_3 + kh^2(\theta_3 - \theta_2) = 0$$

4.3 The Principle of Least Action

For conservative dynamical systems there is a very simple rule in classical mechanics, called the principle of least action, from which the path can be calculated. It says that if the system moved from $\mathbf{q} = \mathbf{x}_1$ at time t_1 to $\mathbf{q} = \mathbf{x}_2$ at time t_2, the path in between would be the one for

which integral of the Lagrangian function is a minimum. In terms of electrical circuits it can be translated to say that the system always changes in a fashion that minimises the difference between the energy stored in the inductors and the energy stored in the capacitors. It is quite surprising that the rule really works. To show that it does, let us calculate the path from the above premise.

There are many cases in which nature follows some other minimisation rule. For example, if a wire is bent into a loop and dipped into a soap solution, a film will form to span the loop that will minimise the area bounded by the wire. If a flexible wire is held at two ends and rotated, it would assume a shape that minimises the surface of revolution. Then there is the famous "brachistochrone" problem invented by Johann Bernoulli, which asks which shape should a frictionless wire have in order that a bead can slide down it in minimum time. In all these cases one has to minimise an integral of a function.

Mathematically the rule can be stated to say that a term (call it S) would be minimised over the path, where

$$S = \int_{t_1}^{t_2} f\,dt. \tag{4.21}$$

In the case of dynamical systems, f = the Lagrangian function $\mathcal{L}$. We first work out the solution for one-dimensional systems for which $\mathcal{L}$ is a function of $q, \dot{q}$ and t. And it is the integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)dt \tag{4.22}$$

that we have to minimise. The problem is not quite the same as the minimisation problems that one handles in elementary calculus. There it is a function that one has to minimise, while here it is an integral. Still, we shall use the same argument which is as follows.

If we have a function q of an independent variable t, the minima has the particular property that if t is varied slightly, the variation in q is negligible (Fig.4.7). In other words, $\frac{dq}{dt} = 0$. This property is not shared by the other points where in general $\triangle q \propto \triangle t$.

How do we carry this argument to paths? Suppose there is a path $q(t)$ that minimises S. If we can vary it somehow by a slight amount, the resulting variation in S should be negligible. To vary the path, we arbitrarily draw a function $\eta(t)$ and obtain a varied path as

$$\bar{q}(t) = q(t) + \alpha.\eta(t) \tag{4.23}$$

where α is a variable quantity. A small α will make $\bar{q}(t)$ deviate slightly from $q(t)$ and a large α will cause a large variation. Thus α becomes the quantity with which we vary S. To obtain the varied path from this formulation, $\eta(t)$ should vanish at the two ends, i.e.,

$$\eta(t_1) = \eta(t_2) = 0 \tag{4.24}$$

We will also need the derivative of (4.23):

$$\dot{\bar{q}}(t) = \dot{q}(t) + \alpha.\dot{\eta}(t) \tag{4.25}$$

We have thus varied the path by α and obtained a new path which has a new S given by

$$S = \int_{t_1}^{t_2} \mathcal{L}(\bar{q}, \dot{\bar{q}}, t)dt \tag{4.26}$$

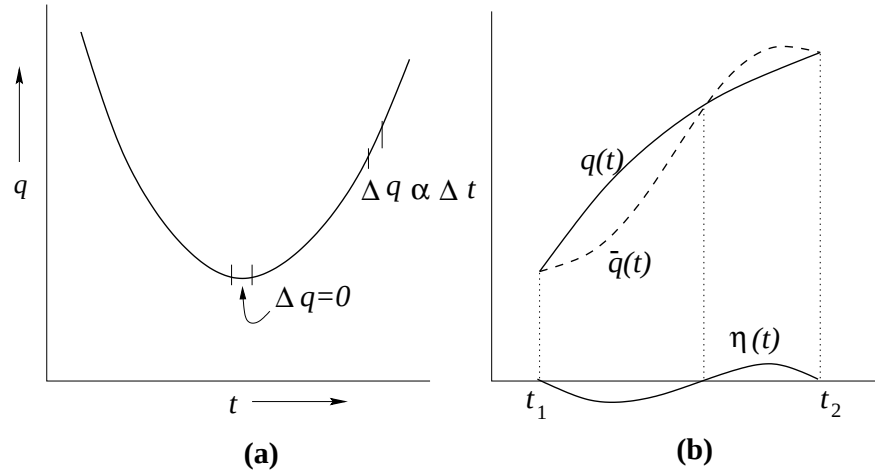


Figure 4.7: (a) The minimal point in a family of points (a function) is obtained by varying t and observing the resulting change in q. (b) The minimal function in a family of functions obtained by varying α and observing the resulting variation in S.

Substituting (4.23) and (4.25) we get

$$S = \int_{t_1}^{t_2} \mathcal{L}(q + \alpha\eta, \dot{q} + \alpha\dot{\eta}, t)dt \tag{4.27}$$

This is a function of α and the condition for minimisation of S is $\frac{\partial S}{\partial \alpha} = 0$ at $\alpha = 0$. Now

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha}\mathcal{L}(\bar{q}, \dot{\bar{q}}, t)dt \tag{4.28}$$

Since $\mathcal{L}$ is a function of several variables, by chain rule we get

$$\frac{\partial}{\partial \alpha}\mathcal{L}(\bar{q}, \dot{\bar{q}}, t) = \frac{\partial \mathcal{L}}{\partial \bar{q}}\frac{\partial \bar{q}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}}\frac{\partial \dot{\bar{q}}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial \alpha} \tag{4.29}$$

Since t is independent of α, $\frac{\partial t}{\partial \alpha} = 0$. The other two terms can be obtained by differentiating (4.23) and (4.25) with respect to α. Thus $\frac{\partial \bar{q}}{\partial \alpha} = \eta(t)$ and $\frac{\partial \dot{\bar{q}}}{\partial \alpha} = \dot{\eta}(t)$. Substituting, we get

$$\frac{\partial}{\partial \alpha}\mathcal{L}(\bar{q}, \dot{\bar{q}}, t) = \frac{\partial \mathcal{L}}{\partial \bar{q}}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}}\dot{\eta}(t) \tag{4.30}$$

Now recall the condition for minimisation of S, which is $\frac{\partial S}{\partial \alpha} = 0$ at $\alpha = 0$. At that value of α, $\bar{q} = q$ and $\dot{\bar{q}} = \dot{q}$.

Substituting this into (4.28) we get

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}(t)\right] dt = 0 \tag{4.31}$$

Integrating the second term by parts we get

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial q}\eta(t)dt + \left.\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta(t)\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta(t)dt = 0 \tag{4.32}$$

By (4.24) the second term would always vanish. Thus the condition for minimum S becomes

$$\int_{t_1}^{t_2} \eta(t)\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]dt = 0 \tag{4.33}$$

Since $\eta(t)$ can be any arbitrary function and the intrgral must vanish for all ηs, the term in bracket must be zero. Hence

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{4.34}$$

This is the Lagrangian equation for one generalized coordinate[2]. Thus we see that the principle of least action yields the same result as Newtonian or Lagrangian mechanics.

Systems with two degrees of freedom

Now let us take a system with two degrees of freedom. What does the principle of least action say for this case? Since the system has two degrees of freedom q_1 and q_2, (4.22) would have the form

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q_1}, q_2, \dot{q_2}, t)dt \tag{4.36}$$

In this case we would define two arbitrary functions $\eta_1(t)$ and $\eta_2(t)$ with the boundary conditions

$$\eta_1(t_1) = \eta_1(t_2) = \eta_2(t_1) = \eta_2(t_2) = 0$$

Then we obtain varied functions $\bar{q_1}$ and $\bar{q_2}$ with the help of a variable α as

$$\begin{aligned} \bar{q_1}(t) &= q_1(t) + \alpha\eta_1(t) \\ \bar{q_2}(t) &= q_2(t) + \alpha\eta_2(t) \end{aligned}$$

The condition for minimum S in this case is the same, that is, $\frac{\partial S}{\partial \alpha} = 0$ at $\alpha = 0$. Following the same procedure we get the condition as

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_1}\eta_1(t) + \frac{\partial \mathcal{L}}{\partial \dot{q_1}}\dot{\eta_1}(t) + \frac{\partial \mathcal{L}}{\partial q_2}\eta_2(t) + \frac{\partial \mathcal{L}}{\partial \dot{q_2}}\dot{\eta_2}(t)\right]dt = 0 \tag{4.37}$$

[2]The general solution to the problem (4.21) yields

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right) - \frac{\partial f}{\partial q} = 0 \tag{4.35}$$

which is known as the Euler's equation. One may notice that the procedure does not ensure that S is a minimum. It could be a maximum or a point of inflection. What the procedure ensures is that the function obtained is *stationary* against variations. The type of extremum is given by the particular problem in hand.

The terms involving $\dot{\eta_1}$ and $\dot{\eta_2}$ are then integrated by parts which yield (after putting the boundary conditions of η_1 and η_2):

$$\int_{t_1}^{t_2} \left\{ \eta_1(t) \left[\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}} \right) \right] + \eta_2(t) \left[\frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}} \right) \right] \right\} dt = 0 \tag{4.38}$$

Since this must hold for all arbitrary choices of the functions $\eta_1(t)$ and $\eta_2(t)$, we get two Euler's equations:

$$\left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}} \right) - \frac{\partial \mathcal{L}}{\partial q_1} \right] = 0 \tag{4.39}$$

$$\left[\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}} \right) - \frac{\partial \mathcal{L}}{\partial q_2} \right] = 0 \tag{4.40}$$

Thus the law of least action yields two Lagrangian equations for q_1 and q_2. Generalizing the line of argument, one can see that for a system with n configuration coordinates, the same least action principle is equivalent to a Lagrangian equation for each of the independent degrees of freedom.

4.4 Lagrangian method applied to electrical circuits

Though the Lagrangian formalism was developed to remove the hurdles in obtaining the differential equations for mechanical systems, the same method can also be applied to obtain the differential equations for electrical circuits. A particular advantage of using the same technique for electrical as well as mechanical systems is that it offers a unified framework for electromechanical systems, i.e., systems with both electrical and mechanical components.

We note, first, that the generalized coordinates in a mechanical system are the position variables that are consistent with the constraints. The constraints in an electrical circuit are given by the way the circuit components are connected, and the electrical equivalent of position variable is the charge. Therefore, our choice of variables should be *charge*, and these have to be defined depending on the circuit connection. We already know that a simple way of doing so is to define the variables as the charges flowing in the meshes. This ensures that the charges flowing in all the branches are defined in terms of the variables chosen. Once the configuration coordinates q_i in an electrical circuit have been defined in the above manner, the $\dot{q_i}$ become simply the mesh currents.

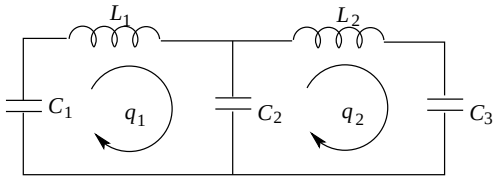


Figure 4.8: An electrical system that is dynamically equivalent to the mechanical system in Fig. 4.5.

The equivalence is illustrated by the circuit shown in Fig.4.8, whose dynamical equations are the same as that of the mechanical system in Fig.4.5. In that sense, we call it the electrical equivalent of the mechanical system.

To show that it indeed has the same dynamics, recall the equivalences between mechanical and electrical systems shown in Chapter 1. An inductor is an inertial element like a mass, a capacitor is a compliant element like a spring. The position coordinate is represented by the charges, and in this circuit, we define the charges flowing in the two loops as the generalized coordinates. In terms of these coordinates, the kinetic energy and the potential energy — and hence the Lagrangian — take exactly the same form as in the mechanical system. This yields exactly the same differential equations. The initial condition may be some energy initially stored in the capacitors which will be equivalent to some initial displacement of the masses in the mechanical system. The dynamics of the two systems would be identical if the corresponding parameters are the same.

It may be noted that the definition of the generalized coordinates is not unique. In fact the charges flowing in any set of branches would suffice so long as the charges in all the branches are given in terms of the chosen set. For example, one can choose the charge flowing in the storage elements in the system plus that in the minimum number of other branches so that the charges in all the branches can be described in terms of the chosen variables. The charges in the meshes only form one such choice which is ensured to be sufficient.

4.5 Systems with external forces or e.m.fs

So far we have considered only conservative systems with no externally applies force or voltage source. If there are external forces or voltage sources present in the system as in Fig.4.9, the same general framework can be used to formulate the differential equations. We only need to incorporate the additional force in the potential function. This can be done by adding a term to V. This new term is external force times the generalized coordinate along which the external force acts, i.e.,

$$V \text{ due to external force} = -F_i q_i$$

This term, when differentiated with respect to that generalized coordinate, will yield the external force. Therefore the generalized force is still obtainable from the potential function as

$$Q_i = -\frac{\partial V}{\partial q_i}.$$

Hence the system is still conservative, and same Lagrangian equation can be applied.

Note the negative sign in this expression which is necessary to set the signs right (because $\mathcal{L} = T - V$).

Example 4.7 *The system in Fig.4.9 is different from that in Fig.4.8 in the addition of the voltage source in the first loop. To account for the source, the potential function would be written as*

$$V = \frac{1}{2C_1}q_1^2 + \frac{1}{2C_3}q_2^2 + \frac{1}{2C_2}(q_1 - q_2)^2 - E.q_1$$

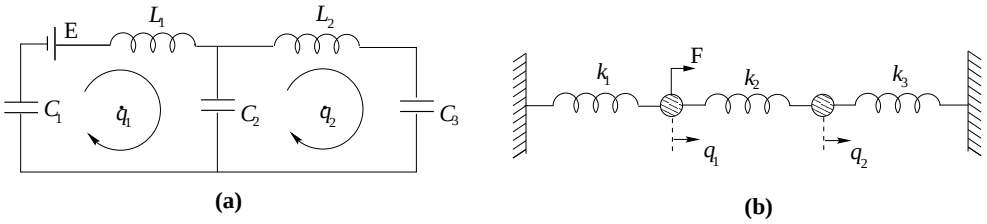


Figure 4.9 A circuit with a voltage source and its mechanical equivalent.

Note the last term coming in to account for the voltage source: the voltage times the coordinate q_1 along which it acts. Thus

$$\frac{\partial \mathcal{L}}{\partial q_1} = -\frac{q_1}{C_1} - \frac{1}{C_2}(q_1 - q_2) + E$$

The first Lagrangian equation will be modified to

$$L_1\ddot{q}_1 + \frac{q_1}{C_1} + \frac{1}{C_2}(q_1 - q_2) - E = 0$$

There will be no change in the second equation.

Example 4.8 *An inverted pendulum with rigid massless rod (length l) is placed on a cart as shown in figure 4.10. The mass of the cart is m_1 and that of the bob is m_2. An external force $F(t)$ is applied on the cart. Derive the dynamical equations of the system.*

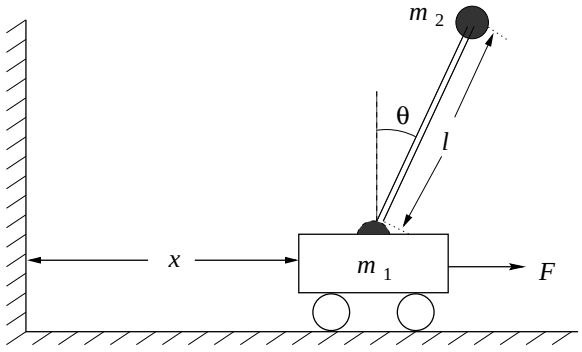


Figure 4.10 Inverted pendulum on cart pertaining to the example.

Solution: *Let the generalized coordinates be x and θ as shown in the figure. The kinetic energy of the cart is $\frac{1}{2}m_1\dot{x}^2$. The kinetic energy of the bob has two components. The horizontal component is $\frac{1}{2}m_2(l\dot{\theta}\cos\theta + \dot{x})^2$ and the vertical component is $\frac{1}{2}m_2(l\dot{\theta}\sin\theta)^2$. The potential also has two parts. One is due to the height of the bob from the level of contact between the rod and the cart. Its value is $m_2gl\cos\theta$. The other is due to the applied force and its value*

is $-Fx$. Thus the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\theta}^2 + m_2l\dot{\theta}\dot{x}\cos\theta - m_2gl\cos\theta + Fx$$

We obtain the partial derivatives as

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}} &= m_1\dot{x} + m_2\dot{x} + m_2l\dot{\theta}\cos\theta \\
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= m_2l^2\dot{\theta} + m_2l\dot{x}\cos\theta \\
\frac{\partial\mathcal{L}}{\partial x} &= F \\
\frac{\partial\mathcal{L}}{\partial\theta} &= -m_2l\dot{\theta}\dot{x}\sin\theta + m_2gl\sin\theta
\end{aligned}$$

Putting these into the Lagrangian equations, we get the equations for x and θ as

$$m_1\ddot{x} + m_2\ddot{x} + m_2l\cos\theta\ddot{\theta} - m_2l\dot{\theta}^2\sin\theta - F = 0$$

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0$$

4.6 Systems with resistance or friction

So far we have considered only conservative systems. In such systems the generalized forces are derivable from a scalar potential. The problem with frictional elements is that the force (or emf) across such an element is proportioal to the velocity (or current). Naturally the force cannot be derived from a potential that is function of the position coordinates.

We would, however, like to retain the advantages of the Newton-Lagrange formalism. In doing so, let us recall the primary form of Lagrangian equation that we derived in (4.12):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i = 0$$

Starting from here we had gone on to include all the possible forms of *conservative forces* and had derived the equation (4.15) involving the Lagrangian function.

In order to extend the formulation to dissipative elements we will have to include the resistive (or frictional) force in the generalized force term Q_i in this original Lagrangian equation. Since this force is velocity dependent, we can achieve this feat by introducing a velocity dependent potential which would give the resistive force upon differentiation by $\dot{q}$. This is called the Rayleigh potential, given by

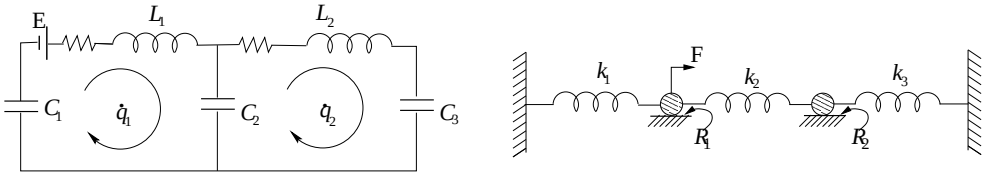
$$\Re = \sum\frac{1}{2}R_i\dot{q}_i^2 \tag{4.41}$$

where R_i are the resistances along the q_i coordinates. Now the generalized force term Q_i would consist of the conservative component and the dissipative component as

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial\Re}{\partial\dot{q}_i}.$$

Thus the equation (4.12) would be modified as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial \Re}{\partial \dot{q}_i} &= 0 \\ \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) - \frac{\partial (T-V)}{\partial q_i} + \frac{\partial \Re}{\partial \dot{q}_i} &= 0 \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \Re}{\partial \dot{q}_i} &= 0 \end{aligned} \tag{4.42}$$



(a) **(b)**

Figure 4.11 A circuit with a resistive element and its mechanical equivalent.

Example 4.9 *Let us consider the electrical circuit in Fig.4.11(a) and the mechanical equivalent system in Fig.4.11(b). The voltage across the resistances (in the mechanical system the forces exterted by frictional elements) are $R_1\dot{q}_1$ and $R_2\dot{q}_2$ respectively. These can be taken into account by defining the Reyleigh potential as*

$$\Re = \frac{1}{2}R_1\dot{q_1}^2 + \frac{1}{2}R_2\dot{q_2}^2 \tag{4.43}$$

So that

$$\frac{\partial \Re}{\partial \dot{q}_1} = R_1\dot{q}_1,$$
$$\frac{\partial \Re}{\partial \dot{q}_2} = R_2\dot{q}_2.$$

Recall from Examples 4.5 and 4.7 that

$$\begin{aligned} T = &\ \tfrac{1}{2}L_1\dot{q_1}^2 + \tfrac{1}{2}L_2\dot{q_2}^2 \quad \textit{in electrical domain} \\ = &\ \tfrac{1}{2}m_1\dot{q_1}^2 + \tfrac{1}{2}m_2\dot{q_2}^2 \quad \textit{in mechanical domain} \end{aligned}$$

$$\begin{aligned} V = &\ \tfrac{1}{2C_1}q_1^2 + \tfrac{1}{2C_2}q_2^2 + \tfrac{1}{2C_3}(q_1-q_2)^2 - E.q_1 \quad \textit{in electrical domain} \\ = &\ \tfrac{1}{2}k_1q_1^2 + \tfrac{1}{2}k(q_1-q_2)^2 + \tfrac{1}{2}k_2q_2^2 - F.q_1 \quad \textit{in mechanical domain} \end{aligned}$$

Applying (4.42), the Lagrangian equation for the q_1 coordinate is

$$L_1\ddot{q}_1 + \frac{q_1}{C_1} + \frac{1}{C_2}(q_1-q_2) - E + R_1\dot{q}_1 = 0$$

Similarly, for the q_2 axis we get

$$L_2\ddot{q}_2 + \frac{q_2}{C_3} - \frac{1}{C_2}(q_1 - q_2) = 0$$

These are the two equations governing the dynamics of the electrical circuit.

It is easy to see that the equations for the mechanical system have similar form, only L is replaced by m and $1/C$ is replaced by k.

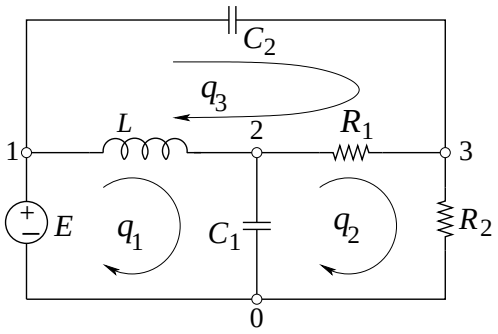


Figure 4.12 The circuit pertaining to Example 4.10.

Example 4.10 *In this circuit there are three meshes, and we define the coordinates as the charges flowing in the meshes as shown in Fig. 4.12. In terms of these variables,*

$$\begin{aligned}
T &= \frac{1}{2}L(\dot{q}_1 - \dot{q}_3)^2, \\
V &= \frac{1}{2C_1}(q_1 - q_2)^2 + \frac{1}{2C_2}q_3^2 - Eq_1, \\
\mathcal{L} &= \frac{1}{2}L(\dot{q}_1 - \dot{q}_3)^2 - \frac{1}{2C_1}(q_1 - q_2)^2 - \frac{1}{2C_2}q_3^2 + Eq_1, \\
\Re &= \frac{1}{2}R_1(\dot{q}_2 - \dot{q}_3)^2 + \frac{1}{2}R_2\dot{q}_2^2.
\end{aligned}$$

From these, we obtain

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_1} = L(\dot{q}_1 - \dot{q}_3), \quad \frac{\partial\mathcal{L}}{\partial q_1} = -\frac{1}{C_1}(q_1 - q_2) + E, \quad \frac{\partial\Re}{\partial\dot{q}_1} = 0.$$

Thus the Lagrangian equation in the q_1 coordinate is

$$L(\ddot{q}_1 - \ddot{q}_3) + \frac{1}{C_1}(q_1 - q_2) - E = 0 \tag{4.44}$$

In the q_2 coordinate,

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_2} = 0, \quad \frac{\partial\mathcal{L}}{\partial q_2} = \frac{1}{C_1}(q_1 - q_2), \quad \frac{\partial\Re}{\partial\dot{q}_2} = R_1(\dot{q}_2 - \dot{q}_3) + R_2\dot{q}_2,$$

and thus the differential equation is

$$-\frac{1}{C_1}(q_1 - q_2) + R_1(\dot{q}_2 - \dot{q}_3) + R_2\dot{q}_2 = 0. \tag{4.45}$$

Similarly, in the q_3 coordinate,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_3} = -L(\dot{q}_1 - \dot{q}_3), \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{1}{C_2}q_3, \quad \frac{\partial \Re}{\partial \dot{q}_3} = -R_1(\dot{q}_2 - \dot{q}_3),$$

and thus the differential equation is

$$L(\ddot{q}_1 - \ddot{q}_3) + \frac{1}{C_2}q_3 + R_1(\dot{q}_2 - \dot{q}_3) = 0. \tag{4.46}$$

The differential equations of the system are given by (4.44), (4.45), and (4.46). A closer scrutiny reveals that these are nothing but the KVL equations in the three meshes, expressed in terms of the chosen coordinates.

4.7 Modeling mutual inductances

In case the inductances in a system have magnetic coupling, leading to mutual inductance effect, it can easily be accounted in terms of a change in the kinetic energy. We know that if there are two coils with self inductances L_1 and L_2, and mutual inductance M, and if currents i_1 and i_2 flow in them, then the total energy stored is

$$T = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2.$$

The sign of the mutual inductance term depends on whether the magnetic coupling is additive or subtractive. If the coils are wound in the same sense, the magnetic fields reinforce each other, resulting in a higher amount of energy storage. In that case we take the positive sign. If, on the other hand, the coils are wound in the opposite sense, the total magnetic field reduces and we have to take the negative sign.

Normally this property is denoted by a dot placed on one of the sides of the inductors. By the dot convention, the magnetomotive forces due to the two currents are additive if both the currents enter (or leave) the respective coils through the ends marked by the dot.

Example 4.11 *In the circuit of Fig. 4.13, there are two loops, and so the coordinates are chosen as q_1 and q_2. In terms of these coordinates,*

$$\begin{aligned}
T &= \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2(\dot{q}_1 - \dot{q}_2)^2 - M\dot{q}_1(\dot{q}_1 - \dot{q}_2) \\
V &= \frac{1}{2C}q_2^2 - Eq_1 \\
\mathcal{L} &= \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2(\dot{q}_1 - \dot{q}_2)^2 - M\dot{q}_1(\dot{q}_1 - \dot{q}_2) - \frac{1}{2C}q_2^2 + Eq_1 \\
\Re &= \frac{1}{2}R_1\dot{q}_2^2
\end{aligned}$$

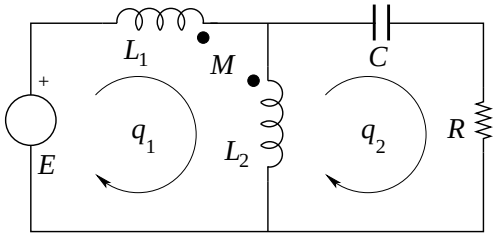


Figure 4.13 The circuit pertaining to Example 4.11.

These give

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = L_1\dot{q}_1 + L_2(\dot{q}_1 - \dot{q}_2) - 2M\dot{q}_1 + M\dot{q}_2, \quad \frac{\partial \mathcal{L}}{\partial q_1} = E, \quad \frac{\partial \Re}{\partial \dot{q}_1} = 0.$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = -L_2(\dot{q}_1 - \dot{q}_2) + M\dot{q}_1, \quad \frac{\partial \mathcal{L}}{\partial q_2} = -\frac{q_2}{C}, \quad \frac{\partial \Re}{\partial \dot{q}_2} = R\dot{q}_2.$$

Thus the two Lagrangian equations become

$$L_1\ddot{q}_1 + L_2(\ddot{q}_1 - \ddot{q}_2) - 2M\ddot{q}_1 + M\ddot{q}_2 - E = 0$$

and

$$-L_2(\ddot{q}_1 - \ddot{q}_2) + M\ddot{q}_1 + \frac{q_2}{C} + R\dot{q}_2 = 0.$$

4.8 A general methodology for electrical networks

In case of electrical circuits we have already seen that the mesh currents form a minimum set of independent variables. The same approach can be followed to identify the generalized coordinates and to formulate the differential equations in terms of these coordinates. The only difference is that in the Lagrangian methodology, the generalized coordinates should be position variables which are equivalent to *charges* in electrical domain. Therefore, the independent generalized coordinates become the charges flowing in the loops.

If there is an electrical network with n "windows" or loops, assign a generalized coordinate q_i to each loop. Thus the loop currents will be $\dot{q}_i$. Write V, T, and $\Re$ in terms of these variables as

$$V = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{C_i}q_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{1}{C_{ij}}(q_i - q_j)^2 - \sum_{i=1}^{n}E_iq_i - \sum_{i=1}^{n}\sum_{j=1}^{i-1}E_{ij}(q_i - q_j) \quad (4.47)$$

where C_i is the capacitance in the ith loop that is not common with other loops; $C_{ij} = C_{ji}$ is the capacitance in the branch common to the ith and jth loops; E_i is the impressed voltage (positive in the positive direction of $\dot{q}_i$) in the ith loop and E_{ij} is the voltage source in the branch common to the ith and jth loops. Expressed in a similar manner we have

$$T = \frac{1}{2}\sum_{i=1}^{n}L_i\dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}L_{ij}(\dot{q}_i - \dot{q}_j)^2 + \frac{1}{2}\sum_{i\neq j}M_{ij}\dot{q}_i\dot{q}_j \quad (4.48)$$

$$\Re = \frac{1}{2}\sum_{i=1}^{n} R_i \dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} R_{ij}(\dot{q}_i - \dot{q}_j)^2 \tag{4.49}$$

where R_i and L_i are the resistance and the coefficient of self inductance in the ith loop not common with any other loop; R_{ij} and L_{ij} are the resistance and the coefficient of self inductance of the branch in common to ith and jth loops. M_{ij} are the mutual inductances between the ith and the jth loops. According to the dot convention, M_{ij}s will be positive if both the currents enter the the dots or come out of the dots (fluxes help each other and hence the stored energy increases), and will be negative if one enters and the other leaves the dots.

The above methodology of obtaining the differential equations follow the mesh current method. It has been shown in Chapter 3 that the mesh current method cannot identify the minimum number of independent variables in case the system has dependent storage elements. This weakness is shared by the Lagrangian method, which yields correct equations in all circumstances, but these may not be the minimal set required to describe the dynamics of a system. This method is therefore apt for systems where there is no closed path containing only capacitances and voltage sources, or a cutset containing only inductances.

4.9 Modeling Coulomb Friction

Coulomb (or dry) friction in mechanical systems poses a problem in modeling because of its nonlinear nature (see Section 1.2.3). Nevertheless, we can approach the problem first by assuming it represented by a resistance element R, and then substituting the nonlinear functional form of R.

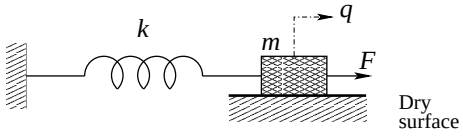


Figure 4.14 The mass-spring-damper system with Coulomb damping.

To illustrate, let us take the simple mass-spring-damper system shown in Fig. 4.14. We derive the Lagrangian equation the usual way:

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 + Fq, \qquad \Re = \frac{1}{2}R\dot{q}^2$$

Which give the Lagrangian equation as

$$m\ddot{q} + kq - F + R\dot{q} = 0$$

Now we note that R can be approximated by a piecewise constant functional form shown in Fig. 4.15. If μ is the coefficient of kinetic friction and $N = mg$ the normal reaction, then the Coulomb friction force is $+\mu N$ for positive values of $\dot{q}$ and $-\mu N$ for negative values of $\dot{q}$

(assuming the force acts in the direction opposite to $\dot{q}$). Since the sign of the friction force depends on the sign of $\dot{q}$, it can be represented by the *signum* function

$$\text{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|} = \begin{cases} +1 & \text{if} \quad \dot{x} > 0 \\ -1 & \text{if} \quad \dot{x} < 0 \end{cases}$$

Therefore the Lagrangian equation becomes

$$m\ddot{q} + kq - F + \mu mg\frac{\dot{x}}{|\dot{x}|} = 0$$

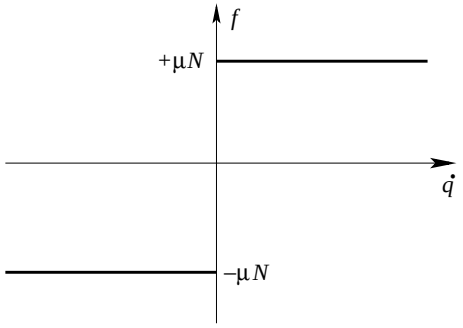


Figure 4.15 The kinetic friction force in Coulomb damping.

4.10 Chapter Summary

The Lagrangian formalism offers a powerful method of obtaining the second-order differential equations for any mechanical, electrical or electromechanical system. To apply this methodology, the steps are:

- Identify the configuration coordinates — the minimum number of variables that specify the positional status of a system.
- Express the potential energy and the kinetic energy as functions of these coordinates and their time derivatives. The potential energy term includes energy stored in capacitors and springs, potential energy due to gravity, and the energy due to any externally applied force or emf (force $\times$ displacement).
- Express the Lagrangian function as $\mathcal{L} = T - V$.
- For nonconservative systems, define a Rayleigh potential as $\sum \frac{1}{2} R_i \dot{q}_i^2$, where $\dot{q}_i$ is the relative velocity at the i-th friction element or the current through the i-th resistance.
- Obtain the partial derivatives of the Lagrangian function and the Rayleigh function with respect to the generalized coordinates and the generalized velcities.

- The system of second order differential equations are then given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \Re}{\partial \dot{q}_i} = 0 \tag{4.50}$$