Terminal Velocity proportional to the Drag Force $m\gamma v^2$ in free fall

$$F(v) = mg - m\gamma v^2$$

$$m\frac{dv}{dt} = mg - m\gamma v^2$$

$$\frac{dv}{dt} = g - \gamma v^2$$

$$\frac{dv}{(g - \gamma v^2)} = dt$$

$$\int_{v(t_0)}^{v(t)} \frac{dv'}{(g - \gamma v'^2)} = \int_{t_0}^{t} dt'$$

$$\int_{v(t_0)}^{v(t)} \frac{dv'}{(g - \gamma v'^2)} = t - t_0$$

$$\int_{v(t_0)}^{v(t)} \frac{dv'}{\gamma(\frac{g}{\gamma} - v'^2)} = t - t_0$$

$$\int_{v(t_0)}^{v(t)} \frac{dv'}{\frac{g}{\gamma} - v'^2} = \gamma(t - t_0)$$

$$\int_{v(t_0)}^{v(t)} \frac{dv'}{\left(\sqrt{\frac{g}{\gamma}}\right)^2 - v'^2} = \gamma(t - t_0)$$

$$\int_{v(t_0)}^{v(t)} \frac{\frac{dv'}{\left(\sqrt{\frac{g}{\gamma}}\right)^2}}{\frac{\left(\sqrt{\frac{g}{\gamma}}\right)^2}{\left(\sqrt{\frac{g}{\gamma}}\right)^2} - \frac{v'^2}{\left(\sqrt{\frac{g}{\gamma}}\right)^2}} = \gamma(t - t_0)$$

$$\frac{1}{\left(\sqrt{\frac{g}{\gamma}}\right)^2} \int_{v(t_0)}^{v(t)} \frac{dv'}{1 - \left(\frac{v'}{\sqrt{\frac{g}{\gamma}}}\right)^2} = \gamma(t - t_0)$$

$$\frac{v'}{\sqrt{\frac{g}{\gamma}}} = u$$

$$\frac{1}{\sqrt{\frac{g}{\gamma}}} dv' = du$$

$$dv' = \sqrt{\frac{g}{\gamma}}.du$$

$$\frac{1}{\left(\sqrt{\frac{g}{\gamma}}\right)^2} \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{du}{1-u^2}.\sqrt{\frac{g}{\gamma}} = \gamma(t-t_0)$$

$$\frac{\sqrt{\frac{g}{\gamma}}}{\left(\sqrt{\frac{g}{\gamma}}\right)^2} \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{du}{1-u^2} = \gamma(t-t_0)$$

$$\frac{1}{\sqrt{\frac{g}{\gamma}}} \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{du}{1-u^2} = \gamma(t-t_0)$$

$$\frac{1}{\sqrt{\frac{g}{\gamma}}} \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{du}{(1-u)(1+u)} = \gamma(t-t_0)$$

Decomposing:

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u} = \frac{A.(1+u)}{(1-u)(1+u)} + \frac{B.(1-u)}{(1+u)(1-u)} = \frac{A+Au+B-Bu}{(1+u)(1-u)}$$

$$\frac{A+B+Au-Bu}{(1+u)(1-u)} = \frac{A+B+u(A-B)}{(1+u)(1-u)}$$

$$\therefore$$

$$\begin{cases} A+B=1 \\ A-B=0 \end{cases}$$

$$\therefore$$

$$A = B$$

$$A + B = 1$$

$$A + A = 1$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$B + B = 1$$

$$2B = 1$$

$$B = \frac{1}{2}$$

$$\therefore$$

$$\frac{1}{\sqrt{\frac{g}{\gamma}}} \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{du}{(1-u)(1+u)} = \gamma(t - t_0)$$

$$\frac{1}{\sqrt{\frac{g}{\gamma}}} \left(\int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{\frac{1}{2}}{(1-u)} du + \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{\frac{1}{2}}{(1+u)} du \right) = \gamma(t - t_0)$$

$$\frac{1}{2\sqrt{\frac{g}{\gamma}}} \left(\int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{1}{(1-u)} du + \int_{\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}}^{\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}} \frac{1}{(1+u)} du \right) = \gamma(t - t_0)$$

$$\frac{1}{2\sqrt{\frac{g}{\gamma}}} \left(\left[\ln\left(1 - \frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right) - \ln\left(1 - \frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right) \right] + \left[\ln\left(1 + \frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right) - \ln\left(1 + \frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right) \right] \right)$$

$$= \gamma(t - t_0)$$

$$\frac{1}{2\sqrt{\frac{g}{\gamma}}} \left(\left[\ln \frac{\left(1 - \frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1 - \frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)} \right] + \left[\ln \frac{\left(1 + \frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1 + \frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)} \right] \right) = \gamma(t - t_0)$$

$$\frac{1}{2\sqrt{\frac{g}{\gamma}}}\left[\ln\frac{\left(1-\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1-\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}.\frac{\left(1+\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1+\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}\right]=\gamma(t-t_0)$$

$$\left[\ln\frac{\left(1-\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1-\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}.\frac{\left(1+\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1+\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}\right]=2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}$$

$$\frac{\left(1-\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1-\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}.\frac{\left(1+\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}\right)}{\left(1+\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}\right)}=e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}$$

$$\frac{1+\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}-\frac{v(t)}{\sqrt{\frac{g}{\gamma}}}-\frac{v^2(t)}{\frac{g}{\gamma}}}{1+\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}-\frac{v(t_0)}{\sqrt{\frac{g}{\gamma}}}-\frac{v^2(t_0)}{\frac{g}{\gamma}}}=e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}$$

$$\frac{1-\frac{v^2(t)}{\frac{g}{\gamma}}}{1-\frac{v^2(t_0)}{\frac{g}{\gamma}}}=e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}$$

$$1-\frac{v^2(t)}{\frac{g}{\gamma}}=\left(1-\frac{v^2(t_0)}{\frac{g}{\gamma}}\right)e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}$$

$$\frac{v^{2(t)}}{\frac{g}{\gamma}}=\left(\frac{v^2(t_0)}{\frac{g}{\gamma}}-1\right)e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}+1$$

$$v^{2(t)}=\left(v^2(t_0)-\frac{g}{\gamma}\right)e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}}+\frac{g}{\gamma}$$

$$v(t) = \sqrt{\left(v^2(t_0) - \frac{g}{\gamma}\right) e^{2\gamma(t-t_0)\sqrt{\frac{g}{\gamma}}} + \frac{g}{\gamma}}$$

Equation with error.