It is equivalent to the following relation:

$$\mu(\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mathbf{r}}{|\mathbf{r}|} + \mathbf{a}, \qquad \mathbf{a} = \text{const}.$$

Consequently,

$$(\mathbf{a}, \mathbf{r}) = |\mathbf{r}|. \tag{1.3}$$

This is the equation of a cone of revolution whose symmetry axis is parallel to the vector $\mathbf{a}$. We demonstrate that the charged particle moves along the geodesics on this cone. Indeed, $\mathbf{r}$ and $\dot{\mathbf{r}}$ are tangent to the cone (1.3). Consequently, the acceleration vector is orthogonal to this cone. Since the speed of motion is constant, by Huygens' formula the normal to the trajectory coincides with the normal to the cone. Therefore the trajectories are geodesics.

This result of Poincaré explains the phenomenon of cathode rays being drawn in by a magnetic pole discovered in 1895 by Birkeland [501].

f) We consider in addition the problem of external ballistics: a material point $(\mathbf{r}, m)$ is moving along a curvilinear orbit near the surface of the Earth experiencing the air resistance. We assume that the resistance force $\mathbf{F}$ has opposite direction to the velocity and its magnitude can be represented in the form

$$|\mathbf{F}| = mg\varphi(v),$$

where φ is a monotonically increasing function such that $\varphi(0) = 0$ and $\varphi(v) \to +\infty$ as $v \to +\infty$.

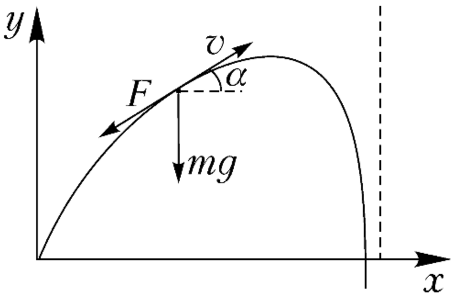

Fig. 1.3. Ballistic trajectory

Since at every moment of time the vectors of the velocity of the point, its weight, and the resistance force lie in the same vertical plane, the trajectory of the point is a planar curve. In the plane of the orbit we introduce Cartesian coordinates x, y such that the y-axis is directed vertically upwards. Let α be the angle between the velocity of the point $\mathbf{v}$ and the horizon (Fig. 1.3). The first of equations (1.2) gives the relation

$$\dot{v} = -g\big[\sin\alpha + \varphi(v)\big]. \tag{1.4}$$

We now make use of the second equation in (1.2). First of all we observe that $\rho = -ds/d\alpha$. The sign "$-$" shows that the angle α decreases as s increases. Taking the projection of the gravitational force onto the normal we arrive at the second relation

$$v\dot{\alpha} = -g\cos\alpha. \tag{1.5}$$

The phase portrait of the closed system of differential equations (1.4) and (1.5) is depicted in Fig. 1.4. All the phase trajectories approach arbitrarily closely the point $\alpha = -\pi/2$, $v = v_0$, where v_0 is the unique positive root of the equation $\varphi(v) = 1$. This point corresponds to the vertical fall of the body with constant velocity (as in example d).

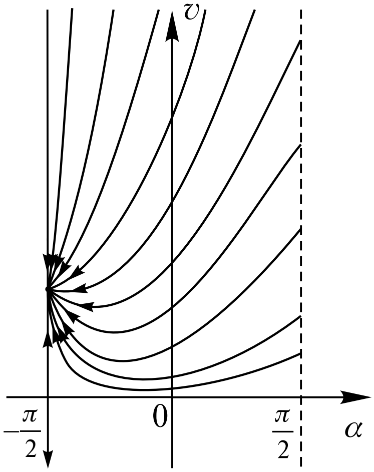

Fig. 1.4. The phase portrait of the ballistic problem

We demonstrate that the trajectory has a vertical asymptote when continued infinitely (as depicted in Fig. 1.3). Indeed, the x-coordinate is determined by the formula

$$x(t) = \int_{t_0}^{t} v\cos\alpha\, dt.$$

We need to show that the corresponding improper integral (when $t = \infty$) converges. For that we pass to a new integration variable α and use (1.5):

$$x = \frac{1}{g}\int_{-\pi/2}^{\alpha_0} v^2\, d\alpha.$$

Since the speed is bounded, this integral has a finite value.

For some laws of resistance, the system of equations (1.4)–(1.5) can be solved explicitly. One of such laws was found already by Legendre:

$$\varphi(v) = cv^\gamma, \qquad c, \gamma = \text{const} > 0.$$

The substitution $u = v^{-\gamma}$ reduces this problem to integrating the single linear differential equation

$$\frac{du}{d\alpha} + \gamma u \tan\alpha + \gamma c \cos^{-1}\alpha = 0.$$

This equation can be easily solved by the method of variation of parameters.

One can find references to other results devoted to the exact integration of equations (1.4)–(1.5), for example, in the book [5].

The principle of determinacy holds also in relativistic mechanics. The difference between classical Newtonian mechanics and relativistic mechanics is in *Galileo's principle of relativity.*

1.1.3 Principle of Relativity

The direct product $E^3 \times \mathbb{R}\{t\}$ (space–time) has the natural structure of an affine space. The *Galilean group* is by definition the group of all affine transformations of $E^3 \times \mathbb{R}$ that preserve time intervals and are isometries of the space E^3 for any fixed $t \in \mathbb{R}$. Thus, if g: $(s, t) \to (s', t')$ is a Galilean transformation, then

1) $t_\alpha - t_\beta = t'_\alpha - t'_\beta$,
2) if $t_\alpha = t_\beta$, then $|s_\alpha - s_\beta| = |s'_\alpha - s'_\beta|$.

The Galilean group obviously acts on $\mathbb{R}^3\{\mathbf{r}\}\times\mathbb{R}\{t\}$. We give three examples of Galilean transformations of this space. First, uniform motion with constant velocity $\mathbf{v}$:

$$g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}t, t).$$

Next, translation of the origin in space–time:

$$g_2(\mathbf{r}, t) = (\mathbf{r} + \mathbf{x}, t + \alpha).$$

Finally, rotation of the coordinate axes:

$$g_3(\mathbf{r}, t) = (G\,\mathbf{r}, t),$$

where $G\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation.

Proposition 1.1. *Every Galilean transformation* $g\colon\ \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ *can be uniquely represented as a composition* $g_1 g_2 g_3$*.*