

Curved Space-Time and the Speed of Light

Anamitra Palit

Author/Teacher, P-154 Motijheel Avenue, Motijheel Housing Cooperative society, Flat-C4, Kolkata-700074, India, Email: palit.anamitra@gmail.com

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ABSTRACT

That the speed of light in vacuum is a universal constant is a well established fact. But is it really so? If two observers stand in curved space-time at two different points and they measure the speed of light at some other common point they are supposed to get different results for the value of 'c' in my opinion. This becomes apparent in the view of the fact that clocks run at different speeds at points having different values of the gravitational potential! Such issues have been investigated in relation to the Schwarzschild Geometry in this article.

1. INTRODUCTION

The constancy of the speed of light^[1] in vacuum is one of the most vital concepts of modern physics. But is it really so? The very question is outrageous enough to be taken cognizance of by the intelligent reader. But I am ready to stand guarantee to the fact that the question is immensely meaningful in relation to curved space-time. I have explored the matter in relation to the Schwarzschild Geometry^[2]. It is a well known fact in General Relativity^[3] that clocks run slow^[4] at places where the gravitational potential has a smaller value, that is, the time intervals are shorter. In regions of higher gravitational potential clocks run faster that is the time intervals are long. (Just think of the fact that the clock hand sweeps out a greater angle between the same pair of events if the clock is running fast).

Let us consider two observers standing at points A and B where the values of gravitational potential are different. And they observe a light ray flashing across a an interval at some point C. The spatial line element is the same for both the observers while the temporal separations are different. So they measure different values for the velocity of light! Now the basic question that arises is that which value corresponds to the speed of light in flat space-time (Minkowski space^[5]). Indeed if a person standing in curved space-time measures the speed of light at the very point where he is standing (with a clock in his hand) he gets the value of 'c' in flat space-time. This is indeed the same value of 'c' one finds in the Schwarzschild metric.

2. SCHWARZSCHILD GEOMETRY REVISITED

The metric for Schwarzschild Geometry is given by,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \text{----- (1)}$$

For constant values of time we have,

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

This is the **spatial** line element which we may denote by $d\Sigma^2$ and write,

$$d\Sigma^2 = (1 - \frac{2GM}{c^2 r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \text{ ----- (2)}$$

Using (2) we may rewrite (1) as,

$$ds^2 = -(1 - \frac{2GM}{c^2 r})(cdt)^2 + d\Sigma^2 \text{ ----- (3)}$$

Now for the path of a light ray (null geodesic) we have

$$a) ds^2 = 0$$

$$b) \frac{d\Sigma}{dT} = c \Rightarrow d\Sigma^2 = c^2 dT^2$$

Using the above results in (3) we may write,

$$ds^2 = -c^2 dT^2 + d\Sigma^2$$

Where dT is the physical time element [$dT = \sqrt{g_{00}} dt$]

For a null geodesic,

$$c^2 dT^2 = d\Sigma^2$$

For two separate space-time points we A and B we have,

$$c^2 dT_A^2 = d\Sigma_A^2$$

And,

$$c^2 dT_B^2 = d\Sigma_B^2$$

The observer measures the same value of the velocity of light at the two points.

3. THE PROBLEM PROPER

But we could have an interesting example where one is standing with a clock at one point and observing a light ray flashing across a spatial interval at some other point.

Now we have,

$$c^2 dT_A^2 = d\Sigma_B^2$$

Or,

$$c^{1/2} dT_B^2 = d\Sigma_A^2$$

The observed velocity of light changes and this does not in any way violate the Special Theory of Relativity!. This is an effect of the curved nature of space-time (and of course not an optical effect).

Now let us calculate the velocity of light at point A as observed by a person standing with a clock at point A and also by a person standing at point B (with his own clock and observing the light ray at A).

The spatial separation (for the passage of the light ray) for both the observers is the same that is,

$$d\Sigma_A^2 = (1 - \frac{2GM}{c^2 r_A})^{-1} dr_A^2 + r_A^2 (d\theta_A^2 + \sin^2 \theta_A d\varphi_A^2)$$

Temporal separation for observer at A is given by,

$$dT_A^2 = -(1 - \frac{2GM}{c^2 r_A})(cdt)^2$$

And the temporal separation for the observer at B is given by,

$$dT_B^2 = -(1 - \frac{2GM}{c^2 r_B})(cdt)^2$$

Speed of light as recorded by A is given by,

$$c_A^2 = \frac{d\Sigma_A^2}{dt_A^2} = \frac{(1 - \frac{2GM}{c^2 r_A})^{-1} dr_A^2 + r_A^2 (d\theta_A^2 + \sin^2 \theta_A d\vartheta_A^2)}{(1 - \frac{2GM}{c^2 r_A})(dt)^2} \quad \text{----- (4)}$$

c_A is identical with the speed of light in flat space-time, that is

$$c_A = c$$

Speed of light as recorded by observer at B is given by,

$$c_B^2 = \frac{d\Sigma_B^2}{dt_B^2} = \frac{(1 - \frac{2GM}{c^2 r_A})^{-1} dr_A^2 + r_A^2 (d\theta_A^2 + \sin^2 \theta_A d\vartheta_A^2)}{(1 - \frac{2GM}{c^2 r_B})(dt)^2} \quad \text{----- (5)}$$

Therefore we have,

$$\frac{c_B^2}{c^2} = \frac{1 - \frac{2GM}{c^2 r_A}}{1 - \frac{2GM}{c^2 r_B}}$$

Now let us take

$$r_A = n \frac{2GM}{c^2}$$

And,

$$r_B = m \frac{2GM}{c^2}$$

We have,

$$\frac{c_B^2}{c^2} = \frac{1 - \frac{1}{n}}{1 - \frac{1}{m}}$$

Or,

$$\frac{c_B^2}{c^2} = \frac{(n-1)m}{(m-1)n} \quad \text{----- (6)}$$

If both the values of m and n are large (and comparable) the above ratio is close to unity. But if one is large while the other is small the ratio is significantly different from one. As an example let us take n=1000 and m=2. Now we have,

$$\frac{c_B^2}{c^2} = 1.998$$

That is,

$$c_B = 1.413c \quad \text{----- (7)}$$

If n=10000 and m=1.001 we have,

$$c_B \approx 31.64c$$

The above calculations clearly bring out the fact that a person standing closer to a black hole will observe a light ray moving in the outward direction with an increasing speed. **Thus the speed barrier is broken.**

Explanation: Let us consider two points A and B, with **B closer to the black hole and A further away from it**, i.e, $n > m$ (conforming to our previous calculations). We consider a **spatial separation at A across which a light ray passes**, noting the fact that in our previous calculations the spatial interval was taken at A. It is the same for both the observers. Time interval is smaller for observer at B and larger for observer at A. Observer at B measures the speed of light to be greater than what A measures. Indeed A measures the value 'c' observed on flat space-time (since he is standing at the very point where the light ray flashes past) and B measures a greater value

We have a significant difference in the value of 'c' and so we need to be careful about our consideration of spherical bodies like black holes. A person standing far away from the Schwarzschild radius will observe a light ray decelerating as it approaches a black hole. We reason out as follows:

Explanation: Let us consider two points A and B, with **A closer to the black hole and B further away from it**, i.e, $n < m$ (conforming to our previous calculations). We consider a **spatial separation at A across which a light ray passes**, noting the fact that in our calculations the spatial interval was taken at A. It is the same for both the observers. Time interval is larger for observer at B and smaller for observer at A. Observer at B measures the speed of light to be smaller than what A measures. Indeed A measures the value 'c' observed on flat space-time (since he is standing at the very point where the light ray flashes past) and B measures a greater value.

Consequently photons would accumulate on approaching a black hole. Decrease in their kinetic energy could lead to further creation of photons leading to increased luminosity.

A person remotely placed from the black hole observes a light ray in retardation as it moves towards the black hole. The velocity of light tends to zero as it approaches the black hole (equation (6) gives $c(B) \rightarrow 0$ as $n \rightarrow 1$). This gives us the impression the nothing should ever fall into a black hole. To get out of the problem we may reason out as follows:

Matter accumulating around a black-hole would change its size increasing the Schwarzschild radius. Thus matter gets engulfed into the black -hole in a peculiar way. This would **increase the surface area** and **reduce the temperature** since the newly added mass consists of low energy particles. This is quite consistent with the existing notions

Now keeping m fixed let us calculate the limit of relation (6) as m tends to infinity.

$$\frac{c_B^2}{c^2} = \lim_{m \rightarrow \infty} \frac{(n-1)m}{(m-1)n}$$

Or,

$$\frac{c_B^2}{c^2} = \lim_{m \rightarrow \infty} \frac{(n-1)}{(1-\frac{1}{m})n}$$

Or,

$$\frac{c_B^2}{c^2} = \frac{n-1}{n}$$

Therefore,

$$c_B = \sqrt{\frac{n-1}{n}}c \quad \text{-----} \quad (8)$$

For small values of n(n=1+ε, where ε is a small fraction) the difference could be quite remarkable!

4.HOW THE PROBLEM APPLIES TO THE EARTH

At points close to the earth's surface we have, $\frac{GM}{c^2 r} \ll 1$. Therefore the Schwarzschild metric could be modified to:

$$ds^2 = -(1 - \frac{2GM}{c^2 r})(cdt)^2 + (1 + \frac{2GM}{c^2 r})dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

We consider observers at A and B observing a light ray flashing across an interval at C. We have,

$$c_A^2 = \frac{d\Sigma_C^2}{dt_A^2} = \frac{(1 + \frac{2GM}{c^2 r_C}) dr_C^2 + r_C^2(d\theta_C^2 + \sin^2 \theta_C d\varphi_C^2)}{(1 - \frac{2GM}{c^2 r_A})(dt)^2} \quad \text{-----} \quad (9)$$

And,

$$c_B^2 = \frac{d\Sigma_C^2}{dt_B^2} = \frac{(1 + \frac{2GM}{c^2 r_C}) dr_C^2 + r_C^2(d\theta_C^2 + \sin^2 \theta_C d\varphi_C^2)}{(1 - \frac{2GM}{c^2 r_B})(dt)^2} \quad \text{-----} \quad (10)$$

Therefore,

$$\frac{c_B^2}{c_A^2} = \frac{1 - \frac{2GM}{c^2 r_A}}{1 - \frac{2GM}{c^2 r_B}}$$

Taking $r_A = n \frac{2GM}{c^2}$ and $r_B = m \frac{2GM}{c^2}$ we have,

$$\frac{c_B^2}{c_A^2} = \frac{(n-1)m}{(m-1)n} \quad \text{-----} \quad (11)$$

For points outside the earth's surface both 'm' and 'n' are very large and hence the above ratio is approximately unity.

5. CONCLUSIONS

It is clear from the above considerations that the measured velocity of light depends on two factors:

- 1) Where the observer is standing with his clock,
- 2) Where the spatial interval of observation is being considered.

Depending on the above two factors different values of the speed of light could be observed in curve space-time. The flat space-time value of 'c' is observed only if the observer observes the light ray at the point where he is standing.

Since the speed of light, $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ the electromagnetic constants are also

affected by the space-time curvature in a relative way! There is an intimate connection between gravitation and electromagnetism.

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