40. Quantization of Gravitational Field: Introduction

We have been discussing the general theory of relativity. Perhaps the three fundamental principles of this theory are the following; i) The events of space-time (all events, past, present, and future) are assembled into a four-dimensional manifold. The description of physics is in terms of fields on this manifold. ii) There is a metric tensor field g_{ab}, on this manifold. The metric simultaneously describes the geometry (results of space and time measurements) of space-time and the effect of gravitation. iii) Matter in space-time produces a certain tensor field which causes the metric of space-time to exhibit curvature. One of the central features of the general theory of relativity (a feature we have, perhaps, not stressed strongly enough) is that the theory claims to incorporate within its structure all of physics. Where there's "physics", there's stress-energy, and hence there's curvature of space-time. The full apparatus of general relativity must, at least in principle (though almost never in practice!) be brought into play in the discussion of any physical phenomenon. General relativity claims a universality over other areas of physics. (Note the word "claims". The entire theory could, of course, differ substantially from the way Nature chooses to behave.)

There exists at least one other theory of physics with a similar claim to universality: quantum theory. In my opinion, the fundamental principles of quantum theory are: i) The states of a system are described in terms of a Hilbert space (more specifically, by rays in a Hilbert space), and ii) the attributes (properties of, measurements on, etc.) of the system are described in terms operators on that Hilbert space. Perhaps quantum theory can be viewed as the insistence that every theory of physics be formulated according to the principles above.

That there is a problem here should now be clear. The general theory of relativity is not formulated in the terms demanded by quantum theory. One seeks, therefore, a modification of the theory to obtain consistency with the principles of quantum theory: one seeks to quantize the general theory of relativity. This new theory should, presumably, not go any more strongly than necessary against the fundamental principles of general relativity. The problem, then, is to write down a theory which is both "quantum-theory-looking" and "general-relativity-looking". This problem, to which a great deal of effort and many clever ideas have been directed, remains unsolved. We shall, in the next few sections, discuss a few of the approaches to this problem. It should be emphasized that whether or not a collection of sentences and equations represents a "solution" to this problem is, for the most part, an aesthetic question.

What features might one expect to appear in a quantum theory of gravitation? It is common that concepts in a classical theory which are "sharp" become "fuzzed out" on quantization. For example, for a particle approaching a potential barrier, classically, the particle either reflects or is transmitted, while, in quantum theory, there is merely a distribution in probabilities for various outcomes. The primary candidate for something to be fuzzed out on quantization of general relativity is the point events of space-time. One might expect that these events will lost their significance – i.e., will reappear only in the classical limit of the quantum theory of gravitation. This expectation is suggested, for example, by the following remark. Suppose we build a probe of some sort which makes measurements in a very small region of space-time (or, in the limit, at a single event of space-time). Then our probe must be at least as small as the region over which it makes measurements. But the uncertainty principle in quantum theory suggests that very small instruments must contain particles of high momentum – hence, high energy. But, if our probe is to have a large stress-energy, then by Einstein's equation, it must be responsible for large curvatures of space-time. In other words, a significant distortion of space-time in the region being measured will result from introducing our probe. Thus, it appears that point events will lose their operational significance under quantization. But it usually happens in physics that, when a concept looses operational significance, that loss is reflected in the mathematical formulation of the theory.

A second concept from general relativity one might expect to be "fuzzed out" by quantization is the metric. There will not, presumably, be one specific metric of space-time, but some probability distribution of possible metrics. This "smearing out of the metric" might be expected to have significant physical consequences. For example, the divergences which arise in quantum field theory come about, at least in part, because integrals in momentum space extend to arbitrary large momenta. One uses "cutoffs" in momentum to obtain finite results. If the metric were "smeared out", one might expect this to result in natural cutoffs on such integrals. One might expect the divergence difficulties associated with quantum field theories to, at least, become less severe in the presence of a quantized metric. Furthermore, the singularities we have seen in general relativity might also be expected to disappear. The smoothing out from quantum theory could result in a smoothing over of these singularities. (Analogous phenomenon in atomic physics: Classically, an electron orbiting a point nucleus radiates, spirals inward, and eventually hits the nucleus. In quantum theory, this singularity disappears.)

The approaches to quantization of general relativity are normally based on analogies with quantum theories we understand: quantum electrodynamics, Schrödinger quantum mechanics for a particle, etc.

41. Linearized Approach to Quantization

Consider the linearized Einstein equations, (166) and (167). We have seen in Sect. 39 that this approach to general relativity results in a set of equations on a field γ_{ab} in flat space which bears a very close resemblance to Maxwell's equations of electrodynamics. The vector potential A_a for the electromagnetic field is replaced by the "potential for the gravitational field", γ_{ab}. These are both tensor fields in flat space-time. In some sense, the linearized Einstein equations represent an approximation to the full equations of general relativity.

The approach to quantization to be discusses in this section is based on the following idea. One regards γ_{ab} as just another classical field (on the same footing with, say, the electromagnetic vector potential). One attempts to use the conventional techniques of quantum field theory on this γ_{ab}. That is to say, one extends the analogy between electrodynamics and linearized general relativity to a quantization program for the latter. Using quantum electrodynamics as a model, one attempts to construct a "quantum gravidynamics".

In this section, we shall first summarize, in very broad and vague terms, the setting of quantum electrodynamics. We then remark that similar techniques could be applied to the linearized Einstein equation. Finally, we make some general comments on the resulting "quantum theory of gravitation".

There are two stages leading to quantum electrodynamics. In the first, one obtains the theory for free photons (the quantized version of the classical theory described by $\nabla^m\nabla_m A^a = 0,\ \nabla_a A^a = 0$). Next, one introduces interactions. For the free case, one proceeds, roughly, as follows. Consider the real (infinite-dimensional) vector space of (asymptotically well-behaved) solutions of Maxwell's equations with $J^a = 0$. One introduces on this vector space a suitable norm and a suitable complex structure. It thus becomes a Hilbert space H. This H represents the Hilbert space of one-photon states of the (source-free) Maxwell field. Next, one extends this description to states with many photons.

$$F = H^0 + H^1 + H^2 + H^3 + \cdots \tag{168}$$

where the superscripts denote "powers" of H. (More precisely, H^0 is the complexes, $H^1 = H$, H^2 is the tensor products of H with itself, H^3 the tensor product of H with H^2, etc. The sums are direct sums of Hilbert spaces.) This F, the Fock space, represents the states of the system (without sources). An element of H^n represents a state with n photons (and an element of H^0 a vacuum (zero photon) state). Thus, the general element of F consists of a linear combination of states with various numbers of photons. There are defined operators on F representing such things as "numbers of photons", "energy-momentum", etc. This, the theory of free (non-interacting) photons, is not very interesting, because nothing much happens.

One now introduces interactions. These are described by certain operators (on F, the Fock space for electrodynamics, and also on the Fock spaces for the other particles of interest, e.g., electrons). These interaction operators allow for the possibility of translations in which numbers of photons change (while, of course, numbers of other types of particles can also change). Thus, with the introduction of an interaction, one has the possibility of particle reactions' taking place. In this way, e.g., electron-photon scattering cross sections can be calculated, and comparison made with experiment.

Essentially the same program goes through, with little change, for the linearized Einstein equation. One introduces the Fock space of free gravitation states. For the interaction, the gravitons couple to the stress-energy of particles rather than (in the electromagnetic case) the charge. One calculates scattering processes, etc. involving gravitons. The proposal, then, is that one regard the result as representing a "quantization of general relativity".

There is certainty a sense in which the program above departs from the spirit of general relativity. One could, of course, criticize it on the grounds that it deals only with the linearized equations – not the full Einstein equation. This, however, is a deficiency only of our brief description – not of the program itself. One could just as well consider also the higher order terms in the perturbation expansion (i.e., in Sect. 39, one could take $d^2/d\lambda^2$, $d^3/d\lambda^3$, etc., at $\lambda = 0$, of Einstein's equation). These corrections would represent further possible interactions – they would be gravitation-gravitation interactions. Thus, one would regard the nonlinearity of Einstein's equation as allowing for the possibility of "gravitational field produced by gravitational field itself". Quantum-mechanically, gravitons create gravitons. The more terms included as interactions from the perturbation expansion, presumably, the closer the resulting field theory would approximate general relativity.

In my view, more serious objections are possible. A physical theory consists, of course, of more than merely the equations of that theory. In particular, general relativity consists of more than Einstein's equation. There is in addition to the equations, an overlay of concepts, attitudes, prejudices, etc. The concepts play at least as great a role in what the theory "is" as the equations. In general relativity, for example, there is the notion of assembling all possible events into a manifold. There is the notion of the metric on this manifold – an object with direct physical significance as giving the result of space and time measurements, and more indirect physical significance concerning gravitation. In short, general relativity is an integral part of what might be called the "space-time view of physics".

Where are these concepts from general relativity in the linearized version of quantized gravitation? One sees, at least, the rudiments of Einstein's equation, but, in my opinion, not the sense of the general theory of relativity. This is not to say, of course, that the linearized program is wrong. What it does seem to imply is that, if Nature behaves as described by this approach, then the general theory of relativity has been an unfortunate – and expensive in terms of time and effort – detour.

42. Canonical Approach to Quantization

As an alternative to the linearized approach, we now discuss the canonical approach to quantization. Perhaps this approach displays a greater respect for the integrity of general relativity than the linearized approach (and, for this reason, it is a good example of an alternative to the linearized). On the other hand, the canonical approach is, it seems to me, a bit on the simple-minded, naive side. It takes, at its model, elementary Schrödinger quantization of a particle. But quantum theory has advanced considerably since its beginnings. We begin with a brief review of Schrödinger quantization. We then attempt to carry over, as directly as possible, these ideas to general relativity. The result is an imprecise, but suggestive, program for obtaining a quantum theory of gravitation.

Consider a single particle. We can describe the particle by its position x and momentum p. Thus, as the particle moves around in time, the motion is described by $x(t)$ and $p(t)$. The dynamics of the particle are described by a pair of differential equations which express $\dot{x}$ and $\dot{p}$ as functions of x and p. Thus, if one specifies the values of x and p at some initial time, then the equations of motion determine x and p for all future times. The initial data for the particle consist of the values of x and p. It usually turns out that the equations of motion for the particle can be cast into the following form. One can find a certain function $H(x, p)$ of x and p such that

$$\dot{x} = \frac{\partial}{\partial p} H(x,p) \qquad \dot{p} = -\frac{\partial}{\partial x} H(x,p) \tag{169}$$

If the equations of motion can be cast into the form (169), they are said to be in *Hamiltonian form*. The function $H(p, q)$ is called the *Hamiltonian* of the system.

The Schrödinger quantization scheme is applicable to classical system whose equations of motion have been placed in Hamiltonian form. The initial data, x, p, are replaced by a single, complex-valued wave function, $\psi(x)$. Instead of $x(t), p(t)$, we have $\psi(x, t)$. Thus, the motion of the system in time is described via time-dependence in ψ. The Hamiltonian equations of motion, (169) are replaced by

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi \tag{170}$$

where $H(x, \frac{\hbar}{i}\frac{\partial}{\partial x})$ means "replace p in $H(x, p)$ by the differential operator $\frac{\hbar}{i}\frac{\partial}{\partial x}$" (a rather vague prescription). Thus, given $\psi(x, t_0)$ for some value of t_0, (170) determines $\psi(x, t)$ for all t.

The Hilbert space for this quantum theory consists of the (complex) vector space of all (sufficiently well-behaved) solutions of (170). One then introduces position, momentum, energy operators, etc.

The idea is, firstly, to try to express the equations of general relativity in "Hamiltonian form". Then, one applies the Schrödinger prescription to obtain a quantum theory. Recall the initial-value formulation of general relativity. The initial data consist of the induced metric h_{ab} and the extrinsic curvature Π^{ab}. These evolve with time according to the equations

$$\begin{aligned} \dot{h}_{ab} &= 2\varphi\Pi_{ab} \\ \dot{\Pi}^{ab} &= -D^a D^b\varphi - 2\varphi\Pi^{am}\Pi^b{}_m - \varphi\Pi\Pi^{ab} + \varphi\mathscr{R}^{ab} \end{aligned} \tag{171}$$

where φ is the evolution function. One first task is to re-express (171) in Hamiltonian form.

Set $p^{ab} = \Pi^{ab} - \Pi h^{ab}$. This p^{ab} is, it turns out, more closely analogous to the p for a particle than Π^{ab}. Rewriting (171) in terms of p^{ab}, we obtain

$$\begin{aligned} \dot{h}_{ab} &= 2(p_{ab} - \frac{1}{2} p\, h_{ab}) \\ \dot{p}^{ab} &= -D^a D^b \varphi + \varphi \mathscr{R}^{ab} - 2\varphi\, p^a{}_m p^{bm} \\ &\quad + \frac{3}{2} \varphi\, p\, p^{ab} - 2\varphi\, h^{ab} p^{mn} p_{mn} + \frac{1}{4} \varphi\, p^2 h^{ab} \end{aligned} \tag{172}$$

We wish to express these equations in Hamiltonian form. This is in fact possible: choose for the Hamiltonian

$$H = -\int_S \varphi\, (\mathscr{R} - p^{mn} p_{mn} + \frac{1}{2} p^2)\, \mathrm{d}V \tag{173}$$

where the integral extends over the entire 3-manifold S. (We ignore questions of convergence of integrals. We shall also allow ourselves to throw away surface terms at will. Such details are unimportant at this stage of theory-building.) Note that $H(h_{ab}, p^{ab})$ does indeed assign a real number to each choice of data, $(h_{ab},\, p^{ab})$, as we would want.

Thus, we have a Hamiltonian formulation of the initial-value formulation of general relativity (ignoring for the moment, the question of constraints). Note that we go to the initial-value formulation of general relativity because, in the one-particle discussion, time played a special role. An analogy could be made to general relativity only reintroducing a "time" there. That is precisely what the initial-value formulation accomplishes. In a certain sense we have, already at this stage, violated the spirit of general relativity.

The next step is to write down the wave function. Instead of the $\psi(x)$ in the Schrödinger theory, we have $\psi(h_{ab})$, a complex-valued function of the collection of all positive-definite metrics on the (fixed) three-dimensional manifold S. We wish to permit evolution, so we write $\psi(h_{ab}, t)$. The Schrödinger equation, (170) becomes, using (173),

$$-\frac{\hbar}{i} \frac{\partial}{\partial t} \psi = \left(\frac{\hbar}{i}\right)^2 \left(h^{ac} h^{bd} - \frac{1}{2} h^{ab} h^{cd}\right) \frac{\delta^2 \psi}{\delta h_{ab}\, \delta h_{cd}} - \mathscr{R}\, \psi \tag{174}$$

where, $\delta/\delta h_{ab}$ refers (rather imprecisely) to functional derivatives. Thus, just carrying over the analogy with Schrödinger quantization, we are led to describe the "quantum gravitational field" by a complex-valued function, $\psi(h_{ab}, t)$, on the space of all positive-definite metrics on S, and on t, This function must satisfy the "gravitational Schrödinger equation", (174).

We have, up till now, ignored the constraint equations, (134) and (135). Clearly, one is doing something essentially wrong if he simply ignores certain equations: we have yet fully incorporated Einstein's equation into our theory. First note that, in terms of p^{ab}, the constraints take the form

$$D_b p^{ab} = 0 \tag{175}$$

$$\mathscr{R} - p^{ab} p_{ab} + \frac{1}{2} p^2 = 0 \tag{176}$$

The question is: How do we "incorporate" these equations into the theory? The most naive answer is simply to incorporate them as operator equations, using the replacement of p_{ab} by $(\hbar/i)\,\delta/\delta h_{ab}$. Let's try this to see what happens. The classical constrain (175) would then be replaced by the following condition on our wave function

$$D_b\left(\frac{\hbar}{i}\frac{\delta\psi}{\delta h_{ab}}\right) = 0 \tag{177}$$

To interpret (177) multiply by an arbitrary vector field v_a on S, and integrate by parts to obtain

$$\int\limits_S \left(\frac{\delta\psi}{\delta h_{ab}}\right)(D_{(a}v_{b)})\,\mathrm{d}V = 0 \tag{178}$$

The validity of (177) is equivalent to the validity of (178) for all v_a. But (178) is easy to interpret. Note that $D_{(a}v_{b)} = \frac{1}{2}\pounds_v h_{ab}$. Thus, $D_{(a}v_{b)}$ is (up to a factor) the rate of change of h_{ab} under the diffeomorphism generated by motions along the integral curves of v_a. Therefore (by the chain rule), (178) states that the rate of change of $\psi(h_{ab})$, as h_{ab} changes by the diffeomorphism generated by v_a, is zero. To say it another way, (178) requires that, if h_{ab} and h'_{ab} are two metrics on S which differ by a diffeomorphism on S (i.e., if h^{ab} and h'^{ab} are isometric), then $\psi(h_{ab}) = \psi(h'_{ab})$. We can write this symbolically as $\psi = \psi(\text{geometry})$. This conclusion is also reasonable physically. The physics of isometric metrics is identical (the only difference being the labeling of points of S). Thus, one might expect ψ to assume the same value on two such metrics. To summarize, the constraint equation (175) leads to the quantum condition (177) which, geometrically, means that ψ is invariant under replacing h_{ab} by the result of subjecting h_{ab} to a diffeomorphism in S.

We now repeat for (176). It is not hard to guess what the answer will be: (176) requires that $\psi(h_{ab}, t)$ be invariant under motions in time. That this is indeed the case can be seen immediately by noting that the Hamiltonian of our theory, (173), is just an integral of the constraint (176). Hence, the quantum version of (176) is precisely the condition that the right side of (174) vanishes. Thus, we require $(\partial/\partial t)\psi(h_{ab}, t) = 0$, i.e., that ψ in fact be independent of t. (It's a good thing. The interpretation of this "t" was always rather obscure, anyway.)

Thus, the constraints, (175) and (176), are related to the section of diffeomorphisms in space-time. This is expressed, in the quantum theory, by certain invariance of the wave function. In fact, we have seen these notions once before: in the gauge transformations in the linearized theory. These gauge transformations, again represented the action of diffeomorphisms. Thus, gauge (linearized version), action of diffeomorphisms (full theory), constraints (initial-value formulation), and invariance of wave function (quantum theory) all are manifestations of essentially the same thing.

We summarize by stating the formalism of this "theory". The Hilbert space is the space of all complex-valued functions $\psi(h_{ab})$ on the space of positive-definite metrics on S, such that ψ is invariant under the action (on h_{ab}) of diffeomorphisms on S, and such that

$$\left(\frac{\hbar}{i}\right)^2\left(h^{ac}h^{bd} - h^{ab}h^{cd}\right)\frac{\delta^2\psi}{\delta h_{ab}\,\delta h_{cd}} - \mathscr{R}\,\psi = 0 \tag{179}$$

We put theory in quotation marks because we have here merely an equation and a few words. What does it all mean? What is the measurement situation? What would it be like to live in such a quantum space-time? What is the correspondence limit? To what extent have the principles of quantum theory been incorporated? To what extent is this theory a "fuzzing out" of general relativity?

What alternative formulations are available, and how do they compare with this one?

We are today a good way from satisfactory answers to questions of this sort.