2310H final 2004 Exam, Smith,

I.(i) If f is a function defined on [a,b] and $a = x_0 \le x_1 \le \ldots \le x_n = b$ is a subdivision of [a,b], describe what an "Riemann sum" for f means, for this subdivision.

(ii) Define what it means for f to be "integrable" on [a,b] in terms of Riemann sums.

(iii) State two essentially different properties, each of which guarantees f is integrable on [a,b].

(iv) Give an example of a function defined, but not integrable, on [0,1].

II. (i) If f is defined by $f(x) = 1/2$ for $0 \le x < 1/2$; $f(x) = 1/4$ for $1/2 \le x < 3/4$; $f(x) = 1/8$ for $3/4 \le x < 7/8$; .....; $f(x) = 1/2^n$ for $(2^{n-1} - 1)/2^{n-1} \le x < (2^n - 1)/2^n$; and $f(1) = 0$, explain why f is integrable on [0,1], and compute the integral. (The FTC is of no use.)

(ii) If f is defined on [0,1] by $1/\mathrm{sqrt}(1+x^4)$, explain why f is integrable, and <u>estimate</u> the integral from above and below. (The FTC is of no use.)

III. Compute the area between the x axis and the graph of $y = \sin^2(x)$, over the interval $[0,\pi]$. (At last the FTC is of use.)

IV. Compute the arclength of the curve $y = (x^2/4) - (\ln(x)/2)$, over the interval $[1,e^2]$.

V. A solid has as base the ellipse $(x^2/25) + (y^2/16) = 1$. If every plane section perpendicular to the x axis is an isosceles right triangle with one leg in the base, find the volume of the solid.

VI. Find the area of the surface generated by revolving the portion of the curve $x^{2/3} + y^{2/3} = 1$ lying in the first quadrant, around the y axis.

VII. Compute the following antiderivatives:

(i) $\displaystyle\int \frac{(2x + e^x)dx}{(x^2 + e^x + 1)^2} =$

(ii) $\int \cot(4x)dx =$

(iii) $\int e^x \cos(x)dx =$

(iv) $\displaystyle\int \frac{\cos(x)}{1 + \sin^2(x)} dx$

VIII. Determine whether the following series converge, and if possible, say explicitly what is the limit. Explain your conclusions.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{[\ln(2)]^n}{n!}$

(ii) 4 - 4/3 + 4/5 - 4/7 + 4/9 - 4/11 ±.................

(iii) $\sum_{n=1}^{\infty} e^{-n}$

(iv) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

IX. Compute the volume generated by revolving the plane region bounded by the x axis and the curve $y = 4 - x^2$, around the line x = 5.

X. Any function f: R^+-->R which is
(i) continuous, (ii) not always zero, and (iii) satisfies f(ax) = f(a) + f(x) for all a, x >0 is a "log" function. Using this, prove that f(x) = $\int_{t=1}^{t=x} \frac{dt}{t}$ is a log function, using appropriate theorems.
[Hint: You will need to show f' exists and then compare the derivatives of f(x) and f(ax).]

XI. We know the only function f such that (i) f is differentiable, (ii) f(0) = 1, and (ii) f' = f, is e^x. Assuming an everywhere convergent power series is differentiable term by term, use the previous fact to prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x. [Hint: First prove it converges everywhere.]

XII. Use the fact that y = tan(x) satisfies the diferential equation y' = 1 + y^2, to find at least the first four terms of the power series for tan(x). Compare the coefficients to what Taylor's formula a(n) = f^(n)(0)/n! gives you.

XIII.
a) If f is a continuous function on the reals, with f(1) = c > 0, what else must be checked to conclude that f(x) = c^x for all x?

b) If a,b are positive numbers, use the method above to prove that the function f(x) = (a^x)(b^x), equals (ab)^x.