APPENDIX 15A: IDENTIFYING THE SUPERPOSITION

The problem to be confronted here is how to distinguish experimentally an ensemble of systems all of which are associated with the same entangled state (15.11), which we begin with by recalling it here:

$$|\text{final}\rangle = \frac{1}{\sqrt{2}}\left[|V, A_{+1}\rangle + |H, A_{-1}\rangle\right] \qquad (15A.1)$$

from an ensemble E_{red} of systems, half of which are associated with state $|V, A_{+1}\rangle$ and the other half with $|H, A_{-1}\rangle$.

We consider various possible processes of measurement and we evaluate the probabilities of their outcomes according to the laws of the theory. Of course, in accordance with the position discussed in the text, in the first case the prescriptions of the standard formalism have to be applied to state (15A.1), with the postulate of the reduction of the wave packet, while in the second case we have to take into account that, in choosing at random a system, it would have a 1/2 probability of being associated with the first and the same probability of being associated with the second of the two states of the ensemble E_{red}. The calculation has then to be carried out, determining the value of the probabilities of the outcomes in the two cases and giving an equal probability to each.

Suppose we carry out a measurement that identifies the position of the pointer on the apparatus. As was shown in Section 15.5, if we use the symbols $P(A = +1)$ and $P(A = -1)$ to indicate the probabilities that the pointer will point to +1 or −1, we find they are equal:

$$P(A = +1) = 1/2, \; P(A = -1) = 1/2. \qquad (15A.2)$$

both for the state (15A.1) and for the ensemble E_{red}. This type of measurement does not therefore enable us to distinguish between the two cases.

Now let us carry out a plane polarization measurement of photons in directions other than vertical. To begin with a precise example, we choose a test of 45° polarization. How should we proceed in order to evaluate the relative probabilities? The analyses of Sections 4.7 and 6.4 tell us that given one state, we ought to express it as a linear combination of normalized states that correspond to possible and mutually exclusive outcomes of the measurement in which we are interested. In our case this means that we ought to express the states of polarizations $|V\rangle$ and $|H\rangle$ in terms of the states $|45\rangle$ and $|135\rangle$, which represent, respectively, the states where the photon passes or fails the test of 45° at polarization. We already know the relations that exist between these states (recall equations [3.2], [2.7], and [2.8]):

$$|V\rangle = \frac{1}{\sqrt{2}}\left[|45\rangle + |135\rangle\right], \quad |H\rangle = \frac{1}{\sqrt{2}}\left[|45\rangle - |135\rangle\right]. \qquad (15A.3)$$

Substituting these formulas into the state (15A.1) we get

$$
\begin{aligned}
|\text{final}\rangle &= \frac{1}{2}\left[|45, A_{+1}\rangle + |135, A_{+1}\rangle + |45, A_{-1}\rangle - |135, A_{-1}\rangle\right] \\
&= \frac{1}{\sqrt{2}}|45\rangle\left\{\frac{1}{\sqrt{2}}\left[|A_{+1}\rangle + |A_{-1}\rangle\right]\right\} + \frac{1}{\sqrt{2}}|135\rangle\left\{\frac{1}{\sqrt{2}}\left[|A_{+1}\rangle - |A_{-1}\rangle\right]\right\}. \quad (15A.4)
\end{aligned}
$$

Note that the states within the curly brackets are normalized states of the measuring apparatus. The equation (15A.4) tells us immediately that the photon states $|45\rangle$ and $|135\rangle$ have coefficients equal to each other and both equal to $1/\sqrt{2}$, which implies that they have a 1/2 probability that the photon overcomes or fails a test for 45.

We now can go on to the case of the ensemble E_{red} and begin by considering a system in the state $|V,A_{+1}\rangle$. This state is now written in such a way as to make clear its decomposition in terms of the states $|45\rangle$ and $|135\rangle$:

$$|V, A_{+1}\rangle = \frac{1}{\sqrt{2}}|45, A_{+1}\rangle + \frac{1}{\sqrt{2}}|135, A_{+1}\rangle, \qquad (15A.5)$$

from which it follows that it has a 1/2 probability of overcoming a test of polarization at 45°. The corresponding formula for the state $|H,A_{-1}\rangle$ is then

$$|H, A_{-1}\rangle = \frac{1}{\sqrt{2}}|45, A_{-1}\rangle - \frac{1}{\sqrt{2}}|135, A_{-1}\rangle, \qquad (15A.6)$$

and this too tells us that a photon like this has a 1/2 probability of passing a test for 45°. Since by choosing a system at random from the ensemble E_{red} this system is with equal probability in the state $|VA_{+1}\rangle$ or in the state $|HA_{-1}\rangle$ and for both of these occurrences it has an equal probability of passing or failing the next test of polarization, it follows that even in this case there is a 1/2 probability that the photon passes or fails a test for polarization at 45°. Once again, this type of measurement does not permit us to distinguish between the state (15A.1) and the statistical mixture E_{red}.

I will not take the time now to show how measurements that have to do exclusively with observables of the apparatus likewise do not permit the distinction. Instead, I will move on to show that measurements of correlation that combine available noncompatible observables with those that characterize either the states of the photons or of the apparatuses in (15A.3), do permit distinguishing the state (15A.1) from the statistical mixture E_{red}.

As far as regards the photons, we have already considered observables incompatible with vertical polarization, that is to say, polarization measurements at 45°. Accordingly, we must substitute the expressions (15A.3) into all the states in which we are interested. We must now consider an observable for the apparatus that is incompatible with the observable that corresponds to the states for which the index is +1 or −1, respectively. Without getting lost in the details about how such a measurement could take place, let us imagine that there exists an observable Z of the apparatus that assumes the value X when the state of the apparatus is

$$|X\rangle = \frac{1}{\sqrt{2}}\left[|A_{+1}\rangle + |A_{-1}\rangle\right] \tag{15A.7}$$

and the value Y when the state is

$$|Y\rangle = \frac{1}{\sqrt{2}}\left[|A_{+1}\rangle - |A_{-1}\rangle\right]. \tag{15A.8}$$

The inverses of these equations are obviously

$$|A_{+1}\rangle = \frac{1}{\sqrt{2}}\left[|X\rangle + |Y\rangle\right], \qquad |A_{-1}\rangle = \frac{1}{\sqrt{2}}\left[|X\rangle - |Y\rangle\right]. \tag{15A.9}$$

The reader will certainly have understood that in the same way in which a measurement of polarization at 45° is incompatible with a measurement of vertical polarization, in a completely analogous way, a measurement of the observable Z is incompatible with one intended to reveal if the pointer of the apparatus points to +1 or −1. In fact, the relationship between the states $|V\rangle,|H\rangle$, and $|45\rangle,|135\rangle$ is exactly the same as the relationship between the states $|A_{+1}\rangle,|A_{-1}\rangle$ and $|X\rangle,|Y\rangle$. We can now proceed in our analysis.

In the case of state (15A.1), by expressing the states $|V\rangle,|H\rangle$ and $|A_{+1}\rangle,|A_{-1}\rangle$ in terms of states $|45\rangle,|135\rangle$ and $|X\rangle,|Y\rangle$, from (15A.3) and (15A.9) we derive

$$|\text{final}\rangle = \frac{1}{\sqrt{2}}\left[|45,X\rangle + |135,Y\rangle\right], \tag{15A.10}$$

which tells us that in a joint measurement of 45 polarization and of the observable Z of the apparatus, the outcome "the photon passes the test" is perfectly correlated with the outcome "X" and alternatively, the outcome "the photon fails the test" is perfectly correlated with the outcome "Y," each of the two pairs of outcomes occurring with an equal probability. On the contrary, the outcomes " the photon passes the test and the apparatus is found in the state Y" or "the photon fails the test and the apparatus is found in the state X" will never happen.

If we move on to consider the ensemble E_{red}, we see that its members will be described either by the state

$$|V,A_{+1}\rangle = \frac{1}{2}\left[|45,X\rangle + |45,Y\rangle + |135,X\rangle + |135,Y\rangle\right], \tag{15A.11}$$

or alternatively by the state

$$|H, A_{-1}\rangle = \frac{1}{2}\left[|45, X\rangle - |45, Y\rangle - |135, X\rangle + |135, Y\rangle\right]. \qquad (15A.12)$$

From equations (15A.11) and (15A.12) we see that in both cases the four pairs of outcomes (45,X), (45,Y), (135,X), and (135,Y) for the joint measurements are equally probable (with a probability of 1/4). It follows that these probabilities are those that characterize the outcomes of the correlation measurements of the indicated observables. We should notice in particular that the pairs of outcomes (45,Y) and (135,X), which can never arise when the state is (15A.1), have the same probability as the other pairs in the case of E_{red}.

This concludes the demonstration that a distinction between the state (15A.1) and the mixture E_{red} is possible, but requires very difficult correlation measurements of observables that are incompatible with those that characterize the terms of (15A.1) itself.