Spectral shifts in general relativity

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A unified approach towards spectral shifts in general relativity brings the cosmological and gravitational redshifts within the same framework as the more familiar Doppler effect. This approach was first proposed by Synge [*Relativity: The General Theory* (North-Holland, Amsterdam 1960)] and is described here in a more simplified form.

I. INTRODUCTION

The change of wavelength of an electromagnetic wave from its value measured at the source to the value measured at the observer is known as spectral shift. Quantitatively we may define it by the parameter

$$z=\frac{\lambda_O-\lambda_S}{\lambda_S}, \tag{1}$$

where λ_S=wavelength at the source and λ_O=wavelength at the observer. It is customary to refer to the cases with $z>0$ as redshifts and those with $z<0$ as blueshifts. This has obvious reference to the shift of spectral lines in the visible part of the spectrum, although in principle (and in practice), Eq. (1) applies to any wavelength range.

In special relativity the most familiar example of spectral shift is the *Doppler effect.* If the source S is moving away from the observer O with a velocity v making an angle θ with the outward radial direction from O to S (see Fig. 1) the formula (1) becomes

$$1+z=\frac{1+(v/c)\cos\theta}{\sqrt{1-(v/c)^2}}=\frac{1+V\cos\theta}{\sqrt{1-V^2}}, \tag{2}$$

where $V=v/c$, c being the speed of light. Henceforth we shall set $c=1$ and refer to V as the velocity of S relative to O.

It is possible to give a relativistically invariant form of Eq. (2) as follows. As shown in Fig. 2, the world lines of O and S are the curves o, s in the space–time diagram. Let us use coordinates x^i, with $x^0=ct$ and $x^\nu(\nu=1,2,3)$ the three Cartesian space coordinates. The line element is

$$ds^2=\eta_{ik}dx^idx^k,\quad \eta_{ik}=\mathrm{diag}(+1,-1,-1,-1), \tag{3}$$

with x_O^i and x_S^i the coordinates of O and S on their world lines.

If the light ray emitted by x_S^i reaches x_O^i, then

$$\eta_{ik}(x_O^i-x_S^i)(x_O^k-x_S^k)=0. \tag{4}$$

Here and in Eq. (3) we are using the summation convention which we shall employ whenever necessary in this paper. Writing the four-velocities of S and O as V_S^i and V_O^i and differentiating Eq. (4) along the world lines with $\Delta\tau_S$, $\Delta\tau_O$ as the elements of proper time along these respective world lines, we get the following relation:

$$\eta_{ik}(x_O^k-x_S^k)(V_O^i\Delta\tau_O-V_S^i\Delta\tau_S)=0.$$

Therefore, the Doppler effect formula becomes

$$1+z=\frac{\Delta\tau_O}{\Delta\tau_S}=\frac{U_{iO}V_S^i}{U_{iO}V_O^i}, \tag{5}$$

where $U_O^i\propto(x_O^i-x_S^i)$ is a tangent vector to the null ray from S to O. We will have occasion to refer to this formula later.

In general relativity we encounter two more examples of spectral shift. In Fig. 3 we have S, a typical point on a spherical massive object of mass M and radius R and O is an external observer at a coordinate radius $r>R$. In this case the gravitational redshift observed by O in the light received from S is given by the formula

$$1+z=\left(1-\frac{2GM}{c^2r}\right)^{1/2}\Big/\left(1-\frac{2GM}{c^2R}\right)^{1/2}. \tag{6}$$

Here the radial coordinate is as given in the Schwarzschild line element. This formula can of course be generalized to other space–time geometries and in essence conveys the fact that redshift occurs in the passage of light from a strong to a weak gravitational field and blueshift occurs for passage in the reverse direction.

The second context in which general relativity describes spectral shift is cosmological. The simplest model of the expanding universe is described by the Robertson–Walker line element:

$$ds^2=dt^2-a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right]. \tag{7}$$

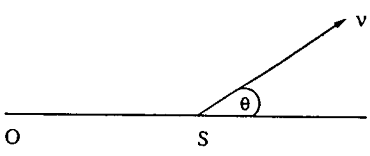

Fig. 1. The Doppler shift formula (2) envisages the source S moving with a speed v in an arbitrary direction making an angle θ with the radial direction OS.

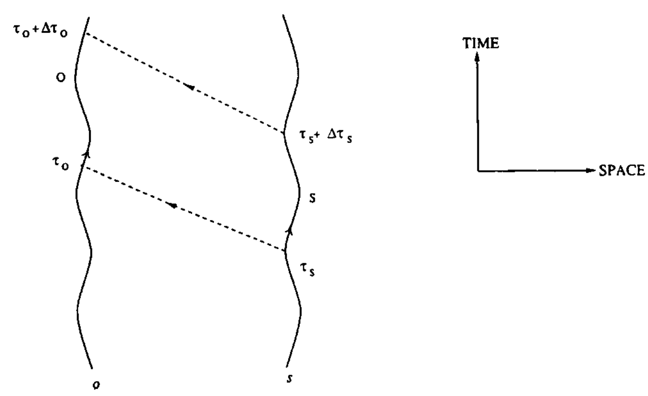

Fig. 2. In the flat space–time diagram o and s are the world lines of the observer O and the source S. The dotted lines are null geodesics representing signal propagation from the source to the observer.

Here the parameter $k=0,\pm 1$ denotes three possibilities for spatial geometry: flat ($k=0$), closed ($k=+1$), and open ($k=-1$). The function $a(t)$ denotes the (expanding) scale factor of the space. The coordinates (r,θ,ϕ) are constant for a typical galaxy and in general define the so-called cosmological rest frame. If S is the galaxy at (r_S,θ_S,ϕ_S) and O is at $r=0$, then the *cosmological redshift* is given by

$$1+z=\frac{a(t_O)}{a(t_S)}, \tag{8}$$

where t_S=epoch of emission of the wave and t_O=epoch of reception. Figure 4 illustrates the scenario.

The way these formulas (1), (6), (8) are derived in most relativity textbooks the reader tends to think of them as different unconnected phenomena. There have been discussions in the literature relating the gravitational redshifts to Doppler shifts in the context of the equivalence principle (see Refs. 1 and 2 for example). However, as first shown by Synge[3] there is an underlying unity behind these concepts which becomes more apparent after the derivations to be described here. We will first work out two specific examples and then prove the theorem which covers the most general scenario.

II. COSMOLOGICAL AND DOPPLER SHIFTS

In the early days after the discovery of the cosmological redshifts and Hubble's law the observers stated the spectral shifts in velocity units, $v=cz$, using the Newtonian Doppler version of Eq. (2). The phrase "recessional velocity" applied to galaxies is still used, although, as seen in Eq. (8) the cosmological spectral shift does not arise from motion.

In fact, in general relativity one cannot talk of a velocity of relative motion between two objects separated spatially without paying due attention to nonlocality and the problem it poses in curved space–time. Thus, given the two world lines (see Fig. 5) o, s of particles with the world points O and S on them connected by a light ray, what do we mean by the velocity of S relative to O? (We assume as in Fig. 5, that the observer O is seeing the source S.). Let V_O^i and V_S^i be the tangent vectors at O and S to the respective world lines, with

$$V_O^i V_{iO}=1, \quad V_S^i V_{iS}=1. \tag{9}$$

Thus in their respective rest frames we have $V_O^0=1$, $V_S^0=1$, as the only nonzero components of velocity. What is the relative velocity of S with respect to O? It is evidently not determined by the special relativistic formula relating to V_S^i and V_O^i since the vector V_S^i at S does not transform as a vector at O and vice versa for V_O^i. Nevertheless, a meaning can be attached to a modified concept. We can parallelly transport V_S^i along the null geodesic SO to O. Let this parallelly transported vector at O be denoted by $\tilde{V}_S^i$. Then we have at O two velocities V_O^i and $\tilde{V}_S^i$ and we can associate a Doppler shift to them. The question is: what answer do we get if S and O are in their respective cosmological rest frames as per the Robertson–Walker space–time?

This question can be answered by solving the relevant equations for parallel transport which we now proceed to do. First we need the affine parameter u along the null geodesic Γ connecting S to O and then set up the differential equation of parallel propagation along it. Along the null geodesic, the condition $ds=0$ leads to (with $c=1$)

$$\frac{dr}{\sqrt{1-kr^2}}=-\frac{dt}{a(t)}. \tag{10}$$

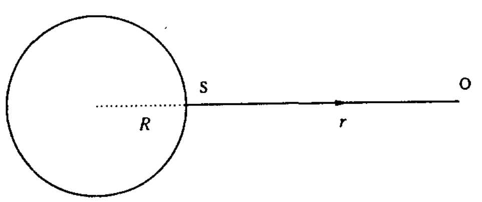

Fig. 3. The source S on the surface of a massive object of radius R emits a signal to a far away observer O. This signal is redshifted due to the gravity of the massive object.

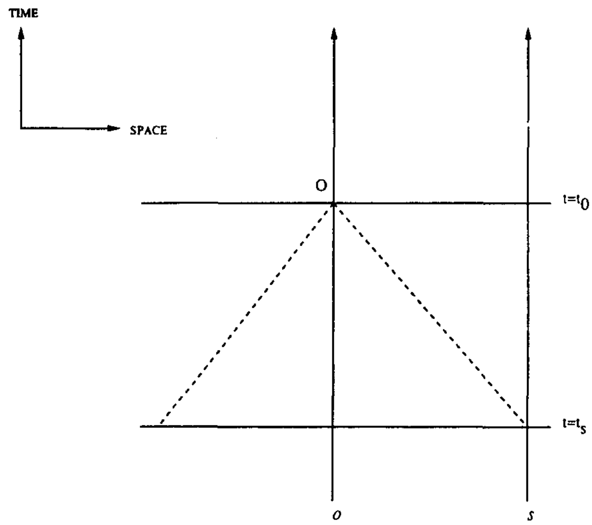

Fig. 4. The past light cone of observer O at epoch t_O intersects the world line of the source S at epoch t_S. The cosmological redshift arises because in an expanding universe the scale factor satisfies the inequality $a(t_O)>a(t_S)$.

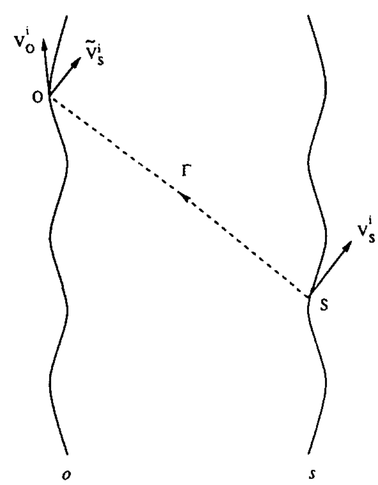

Fig. 5. The velocity vector at S, V_S^i is parallelly transported along the null geodesic Γ from S to O where it takes the value $\tilde{V}_S^i$. The spectral shift phenomenon can be described in terms of the Doppler effect between the velocity vectors V_O^i and $\tilde{V}_S^i$.

To determine u, we use the timelike component of the geodesic equations

$$\frac{d^2x^i}{du^2}+\Gamma^i_{kl}\frac{dx^k}{du}\frac{dx^l}{du}=0, \quad i=0,1,2,3. \tag{11}$$

For notation see Narlikar.[4] The coordinates x^i are, respectively, t, r, θ, ϕ. After computing Γ^i_{kl} we finally find that for $i=0$, Eq. (11) gives

$$\frac{d^2t}{du^2}-\frac{a\dot{a}}{1-kr^2}\left(\frac{dr}{du}\right)^2=0. \tag{12}$$

Using Eq. (10) to eliminate (dr/du), we get a first integral of Eq. (12) as

$$\frac{1}{a}\frac{dt}{du}=\text{constant } A \text{ (say)}. \tag{13}$$

Therefore, from Eq. (10)

$$\frac{1}{\sqrt{1-kr^2}}\frac{dr}{du}=-A. \tag{14}$$

This determines u as a function of t. It is convenient to fix u so that $u=0$ at S and $u=1$ at O. Thus the constant A is given by

$$A=\int_{t_S}^{t_O}\frac{dt}{a(t)}. \tag{15}$$

We define the tangent vector to Γ by $dx^i/du\equiv U^i(u)$, say. Thus from Eq. (14) we get,

$$U^i(u)=A[a(t),-\sqrt{1-kr^2},0,0], \tag{16}$$

with $U^i(1)=A[a(t_O),-1,0,0]$.

Let the vector V^i_S become $V^i(u)$ at an intermediate point u, with $V^i(1)\equiv\tilde{V}^i_S$. Then under parallel propagation

$$V^i(u)V_i(u)=\text{constant}, \quad U^i(u)V_i(u)=\text{constant}. \tag{17}$$

We therefore have the following relations [keeping in mind the fact that $V^i(u)$ has only the t, r components nonzero]

$$\tilde{V}^i_S\tilde{V}_{iS}=1, \quad U^i_O\tilde{V}_{iS}=U^i_SV_{iS}. \tag{18}$$

These relations become

$$(\tilde{V}^0_S)^2-a^2(t_O)(\tilde{V}^1_S)^2=1, \tag{19}$$

$$a(t_O)\tilde{V}^0_S+a^2(t_O)\tilde{V}^1_S=a(t_S). \tag{20}$$

Now, in the local inertial rest frame at O, the velocity vector $\tilde{V}^i_S$ takes the form $(\gamma,\gamma V,0,0)$ where V is the radial three-velocity and $\gamma=(1-V^2)^{-1/2}$. How is V related to $\tilde{V}^1_S$? Since the radial proper distance at O is $|a(t_O)dr|$, and the radially outward direction from S is radially inwards at O, the above relation is

$$\gamma V=-a(t_O)\tilde{V}^1_S. \tag{21}$$

Thus the relations (19) and (20), with Eq. (8) for $a(t_O)/a(t_S)$ and Eq. (21) above for V give the result

$$1+z=\sqrt{\frac{1+V}{1-V}}. \tag{22}$$

We therefore find that the parallelly transported velocity vector of a distant source does yield the correct Doppler velocity in the rest frame of the observer. The relation (2) for radial recession therefore agrees with the cosmological redshift formula (8).

We next look at the gravitational versus the Doppler redshifts.

III. THE GRAVITATIONAL VERSUS DOPPLER SHIFTS

The expanding universe is a dynamical concept and hence the previous result, though nontrivial is not unexpected. The gravitational redshift, however, arises in a manifestly static line element. Hence it is not clear what the corresponding situation would be in this case. Specifically, the formula (6) arises from the line element

$$ds^2=e^\nu dt^2-e^{-\nu}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{23}$$

where, with $c=1$, $x^0=t$, $x^1=r$, $x^2=\theta$, $x^3=\phi$ we have

$$e^\nu=1-\frac{2GM}{r} \quad \text{for } r\geqslant R. \tag{24}$$

We now have $V^i_O=(e^{-\nu_O/2},0,0,0)$, $V^i_S=(e^{-\nu_S/2},0,0,0)$, with $\nu_O=\nu(r_O)$, $\nu_S=\nu(r_S)$ with $r_O>r_S\geqslant R$. [In formula (6) we took $r_S=R$ for light coming from the surface of the massive object.] In this notation the formula (6) is

$$1+z=e^{(\nu_O-\nu_S)/2}. \tag{25}$$

Along the null geodesic Γ connecting S and O, we have, the tangent vector $U^i\equiv dx^i/du$ with the only nonzero components satisfying the relations

$$e^\nu\frac{dt}{du}=\frac{dr}{du}=A \text{ (constant)}, \tag{26}$$

u being the affine parameter chosen so that $u=0$ at S and $u=1$ at O. This determines A completely as

$$A=(r_O-r_S). \tag{27}$$

Let V^i_S, transported parallel to itself along Γ, have the value $V^i(u)$ at any u in the range $0\leqslant u\leqslant 1$. Let $V^i(1)=\tilde{V}^i_S$. Then, the unit magnitude of $\tilde{V}^i$ gives

$$(\tilde{V}^0_S)^2e^{\nu_O}-(\tilde{V}^1_S)^2e^{-\nu_O}=1. \tag{28}$$

Similarly, constancy of $U^i(u)V_i(u)$ gives

$$\tilde{V}^0_S-\tilde{V}^1_Se^{-\nu_O}=e^{-\nu_S/2}. \tag{29}$$

Again, we interpret $\tilde{V}^i_S$ as a four velocity $(\gamma,\gamma V,0,0)$ in the local inertial rest frame at O. Since the proper radial distance is given by $|e^{-\nu_O/2}dr|$ and the proper time by $|e^{\nu_O/2}dt|$ we have

$$V=-e^{-\nu_O}\frac{\tilde{V}^1_S}{\tilde{V}^0_S} \tag{30}$$

[negative sign because as we found for Eq. (21) the radially outward direction from S is inwards at O.]

Write Eq. (29) in the form

$$\tilde{V}^0_Se^{\nu_O/2}-\tilde{V}^1_Se^{-\nu_O/2}=(1+z) \tag{31}$$

and divide Eq. (28) by Eq. (31) to get

$$\tilde{V}^0_Se^{\nu_O/2}+\tilde{V}^1_Se^{-\nu_O/2}=(1+z)^{-1}. \tag{32}$$

It is easy to see that the relations (30)–(32) together give

$$V=\frac{(1+z)-(1+z)^{-1}}{(1+z)+(1+z)^{-1}}, \tag{33}$$

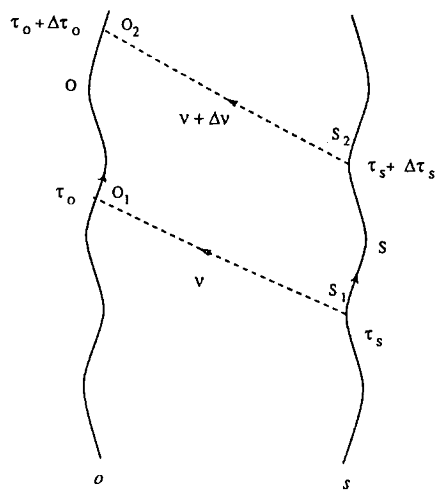

Fig. 6. This is a generalization of Fig. 2 to arbitrary Riemannian space–times. Thus the dotted lines are null geodesics in a general Riemannian space–time.

again giving the Doppler formula applied to the parallelly transported source velocity

$$1+z=\sqrt{\frac{1+V}{1-V}}. \tag{34}$$

IV. THE GENERAL RULE

We have thus established the equivalence of the cosmological and gravitational spectral shifts with the Doppler one provided we parallelly transfer the source four-velocity vector along the null geodesic to the observer. We now show that these results are special cases of a general rule.

To see the unifying theme we refer to the textbook on general relativity by Synge[1] which discusses these issues at length. Figure 6 illustrates the general scenario for spectral shifts. Here o and s are two world lines of observer O and source S, respectively. Suppose further that the passage of light signals from S to O is described by a series of null geodesics connecting their respective world lines. We will use a parameter ν to label a typical null geodesic. Let S_1 and S_2 be two neighboring world points on s and O_1 and O_2 the corresponding world points on o where the null geodesics from S_1 and S_2 meet it. We will assume the parametric values for these geodesics to be $\nu,\nu+\Delta\nu$, respectively, where $\Delta\nu$ is infinitesimally small. Let u denote the affine parameter on each of these geodesics chosen so that $u=0$ on s and $u=1$ on o. Also we will denote by τ_O and τ_S the proper times of the observer and the source, respectively.

Let V_S^i and V_O^i be the respective four-velocity vectors of the source and the observer at S_1 and O_1, respectively. Then

$$V_S^i=\left.\frac{dx^i}{d\tau_S}\right|_{S_1}, \quad V_O^i=\left.\frac{dx^i}{d\tau_O}\right|_{O_1} \tag{35}$$

are the tangent vectors to s and o at these respective points.

Now let $\Delta\tau_O$ and $\Delta\tau_S$ be the elements of proper time corresponding to the segments O_1O_2 and S_1S_2. Then the spectral shift z is given by

$$1+z=\frac{\Delta\tau_O}{\Delta\tau_S}. \tag{36}$$

Following Synge we define a world function for a pair of points O and S through an integral defined along the geodesic joining them

$$I(\nu)=\frac{1}{2}(u_O-u_S)\int_{u_S}^{u_O} g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial u}\,du. \tag{37}$$

$I(\nu)$ is thus defined for any of the geodesics in the family linking points on o and s. We have taken $u_S=0$, $u_O=1$ so that

$$I(\nu)=\frac{1}{2}\int_0^1 g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial u}\,du. \tag{38}$$

Synge has proved the following relations:

$$\left.\frac{\partial I}{\partial x^i}\right|_S=\left.g_{ij}\frac{\partial x^j}{\partial u}\right|_S, \quad \left.\frac{\partial I}{\partial x^i}\right|_O=-\left.g_{ij}\frac{\partial x^j}{\partial u}\right|_O, \tag{39}$$

which hold even for null geodesics. Now since the geodesics S_1O_1 and S_2O_2 are null, the quantity $I(\nu)$ does not change between ν and $\nu+\Delta\nu$. Hence, we have

$$\left.\frac{\partial I}{\partial x^i}\frac{dx^i}{d\nu}\right|_O+\left.\frac{\partial I}{\partial x^i}\frac{dx^i}{d\nu}\right|_S=0,$$

i.e.,

$$U_{iO}V_O^i\Delta\tau_O-U_{iS}V_S^i\Delta\tau_S=0. \tag{40}$$

The U_{iS} and U_{iO} are the tangent vectors to the typical null geodesic connecting S to O at its respective end points. Using Eq. (36) therefore we get

$$1+z=\frac{U_{iS}V_S^i}{U_{iO}V_O^i}. \tag{41}$$

Further, since, along a geodesic the scalar product of the tangent vector and a parallelly propagated vector is constant, we may transport V_S^i parallelly to O where it takes the form $\tilde{V}_S^i$ and thus obtain from Eq. (41),

$$1+z=\frac{U_{iO}\tilde{V}_S^i}{U_{iO}V_O^i}. \tag{42}$$

However, Eq. (5) of special relativity tells us that this is the Doppler shift of a source with four-velocity $\tilde{V}_S^i$ observed by an observer with the 4 velocity V_O^i, with U_{iO} the direction of the null vector connecting the source to the observer as measured by the latter. This establishes the general result stated earlier.

V. CONCLUSION

In his book (op. cit.) Synge has emphasized the unity behind these redshifts. We quote his remarks from Chap. III, Sec. 8 (pp. 122 and 123):

"It is clear that in general the observer of a luminous source will see a spectral shift··· In attributing a *cause* to this spectral shift, one would say, ··· that the spectral shift was caused by the relative velocity of source and observer; it is in fact a Doppler effect in the original sense of the term. It is not a gravitational effect, because the Riemann tensor appears nowhere in our formulae."

In general relativity gravitational effects (as seen near a massive object or in cosmology) are inextricably mixed with local motions. As Synge observes, a truly gravitational effect should involve the Riemann tensor, which the formulas (41) or (42) do not. Thus, in principle, we could have $z \neq 0$ from these formulas even in flat space–time scenarios. For example, the Robertson–Walker model with $k=-1$ and $a(t) \propto t$ is flat but has a cosmological redshift.

This result derived by Synge is, unfortunately, not well known even among the community of general relativists today. With spectral shifts being used so often in different contexts, it is worth appreciating the underlying theme that Synge had highlighted. The purpose of this communication was to bring the result to the notice of modern workers in relativity with the help of explicit examples familiar to them.

[1]N. T. Bishop and P. T. Landsberg, "Equivalence principle: 60 years of a misuse?," Nature **252**, 459–460, (1974)

[2]N. T. Bishop and P. T. Landsberg, "Gravitational redshift and the equivalence principle," Found. Phys. **6**, 727–737 (1976).

[3]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).

[4]J. V. Narlikar, *Introduction to Cosmology* (Cambridge University Press, Cambridge, U.K., 1993).

Dynamic interpretation of Maxwell's equations

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Conventional discussions of Maxwell's equations in free space have for many years taken a historical approach starting with electrostatics and magnetostatics, and have taught us that the sources of **E** are electric charge and $\dot{\mathbf{B}}$, and the sources of **B** are electric current and $\dot{\mathbf{E}}$. However, a direct dynamic reading of Maxwell's differential equations leads unquestionably to the surprisingly different conclusions that the sources of **E** are electric current and curl **B**, and the single source of **B** is curl **E**. In this dynamic reading of Maxwell's equations, electric field is generated locally by electric current, and fields propagate away from the current source by the dual mechanisms of curl **E** generating **B** locally and curl **B** generating **E** locally.

I. INTRODUCTION

Conventional discussions of Maxwell's differential equations in free space

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}, \tag{1}$$

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \dot{\mathbf{E}}), \tag{2}$$

follow the historical development of electromagnetism, proceeding from electrostatics (Coulomb) and magnetostatics (Ampere and Biot–Savart) to Faraday's induction and finally to Maxwell's displacement current and field propagation. It seems to follow naturally from electrostatics and magnetostatics, that charge and current are the sources, respectively, of electric and magnetic field. It again seems natural to interpret the contributions of Faraday and Maxwell by saying that electric field is generated also by time varying magnetic field, and that magnetic field is generated also by time varying electric field. Finally, recognizing the neat fit between these interpretations and the mathematics of the Helmholtz theorem, which shows that a vector field is determined by its divergence and its curl as sources of the field, it is no wonder that most physicists trained in this tradition have had no reason to question these "natural" teachings, or to look for alternative interpretations. The Helmholtz theorem[1]

$$\mathbf{V}(\mathbf{r},t) = \int \operatorname{div} \mathbf{V}(\mathbf{r}',t) \frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}' + \int \operatorname{curl} \mathbf{V}(\mathbf{r}',t) \times \frac{\mathbf{r}-\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}' \tag{3}$$

expresses a vector field $\mathbf{V}(\mathbf{r},t)$ as a sum of an irrotational (Coulomb-type) field with source density div **V**, and a solenoidal (Biot–Savart-type) field with source density curl **V**. The integrals in Eq. (3) extend over all space.

In this paper we describe a surprisingly different interpretation, one that follows naturally by reading Maxwell's differential equations directly as a set of local dynamic field equations. In this reading, the instantaneous state of the electromagnetic field is described by the values of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ at all points of space, and the rate of change of the state, described by $\dot{\mathbf{E}}$ and $\dot{\mathbf{B}}$, is determined by the instantaneous values of the fields and of the current distribution **j**, through the two curl equations of Maxwell,

$$\dot{\mathbf{E}} = -\frac{1}{\epsilon_0} \mathbf{j} + \frac{1}{\epsilon_0 \mu_0} \operatorname{curl} \mathbf{B}, \tag{4}$$

$$\dot{\mathbf{B}} = -\operatorname{curl} \mathbf{E}. \tag{5}$$

We can imagine numerically integrating these equations, time step by time step, using

$$\mathbf{E}(\mathbf{r},t+\delta t) = \mathbf{E}(\mathbf{r},t) + \mathbf{S}_E(\mathbf{r},t)\,\delta t,$$

$$\mathbf{B}(\mathbf{r},t+\delta t) = \mathbf{B}(\mathbf{r},t) + \mathbf{S}_B(\mathbf{r},t)\,\delta t,$$