2.5.3. GVD through Angular Dispersion—General

Angular dispersion has been advantageously used for a long time to resolve spectra or for spectral filtering, utilizing the spatial distribution of the frequency components behind the dispersive element (e.g., prism, grating). In connection with fs optics, angular dispersion has the interesting property of introducing GVD. At first glance this seems to be an undesired effect. However, optical devices based on angular dispersion, which allow for a continuous tuning of the GVD can be designed. This idea was first implemented in Treacy [28] for the compression of chirped pulses with diffraction gratings. The concept was later generalized to prisms and prism sequences [29]. Simple expressions for two and four prism sequences are given in [30, 31]. From a general point of view, the diffraction problem can be treated by solving the corresponding Fresnel integrals [28,32,33]. We will sketch this procedure at the end of this chapter. Another successful approach is to analyze the sequence of optical elements by ray–optical techniques and calculate the optical beam path P as a function of Ω. From our earlier discussion we expect the response of any linear element to be of the form:

$$R(\Omega)e^{-i\Psi(\Omega)} \tag{2.71}$$

where the phase delay Ψ is related to the optical pathlength P_{OL} through

$$\Psi(\Omega) = \frac{\Omega}{c}P_{OL}(\Omega). \tag{2.72}$$

$R(\Omega)$ is assumed to be constant over the spectral range of interest and thus will be neglected.

We know that nonzero terms $[(d^n/d\Omega^n)\Psi \neq 0]$ of order $n > 2$ are responsible for changes in the complex pulse envelope. In particular

$$\frac{d^2}{d\Omega^2}\Psi(\Omega) = \frac{1}{c}\left(2\frac{dP_{OL}}{d\Omega} + \Omega\frac{d^2P_{OL}}{d\Omega^2}\right) = \frac{\lambda^3}{2\pi c^2}\frac{d^2P_{OL}}{d\lambda^2} \tag{2.73}$$

is related to the GVD parameter. We recall that, with the sign convention chosen in Eq. (2.71), the phase factor Ψ has the same sign as the phase factor $k_\ell L$. Consistent with the definition given in Eq. (1.117) a positive GVD corresponds to $\frac{d^2\Psi}{d\Omega^2} > 0$. In this chapter, we will generally express $\frac{d^2\Psi}{d\Omega^2}$ in fs^2.

The relation between angular dispersion and GVD can be derived through the following intuitive approach. Let us consider a light ray which is incident onto an optical element at point O, as in Figure 2.22. At this point we do not specify the element, but just assume that it causes angular dispersion. Thus, different spectral components originate at O under different angles, within a cone represented by the patterned area in the figure. Two rays corresponding to the center frequency ω_ℓ of

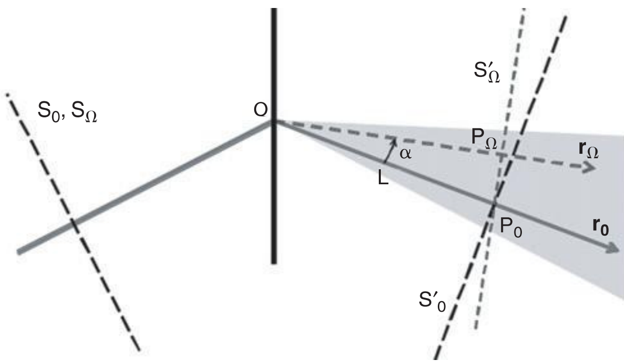

Figure 2.22 Angular dispersion causes GVD. The solid line in the middle of the figure represents the angular dispersive element, providing a frequency-dependent deflection of the beam at the point of incidence *O*. The different frequency components of the pulse spread out in the patterned area.

the spectrum, $\vec{r}_0$, and to an arbitrary frequency Ω, $\vec{r}_\Omega$, are shown in Fig. 2.22. The respective wavefronts S are labelled with subscript "0" (for the central frequency ω_ℓ) and "Ω" (for the arbitrary frequency Ω). The planes S_Ω, S_0 and S'_Ω, S'_0 are perpendicular to the ray direction and represent (plane) wave fronts of the incident light and diffracted light, respectively. Let P_0 be our point of reference and be located on $\vec{r}_0$ where $\overline{OP_0} = L$. A wavefront S'_Ω of $\vec{r}_\Omega$ at P_Ω is assumed to intersect $\vec{r}_0$ at P_0. The optical pathlength $\overline{OP_\Omega}$ is thus

$$\overline{OP_\Omega} = P_{OL}(\Omega) = P_{OL}(\omega_\ell)\cos\alpha = L\cos\alpha \tag{2.74}$$

which gives for the phase delay

$$\Psi(\Omega) = \frac{\Omega}{c}P_{OL}(\Omega) = \frac{\Omega}{c}L\cos\alpha \tag{2.75}$$

The dispersion constant responsible for GVD is obtained by twofold derivation with respect to Ω:

$$\boxed{\begin{aligned} \left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} &= -\frac{L}{c}\left[\sin\alpha\left(2\frac{d\alpha}{d\Omega} + \Omega\frac{d^2\alpha}{d\Omega^2}\right) + \Omega\cos\alpha\left(\frac{d\alpha}{d\Omega}\right)^2\right]_{\omega_\ell} \\ &\approx -\frac{L\omega_\ell}{c}\left(\left.\frac{d\alpha}{d\Omega}\right|_{\omega_\ell}\right)^2 \qquad (2.76) \end{aligned}}$$

where $\sin\alpha = 0$ and $\cos\alpha = 1$ if we take the derivatives at the center frequency of the pulse, $\Omega = \omega_\ell$. The quantity $(d\alpha/d\Omega)|_{\omega_\ell}$, responsible for angular dispersion, is a characteristic of the actual optical device to be considered. It is interesting to note that the dispersion parameter is always negative independently of the sign of $d\alpha/d\Omega$ and that the dispersion increases with increasing distance L from the diffraction point. Therefore angular dispersion always results in negative GVD. Differentiation of Eq. (2.76) results in the next higher dispersion order

$$\left.\frac{d^3\Psi}{d\Omega^3}\right|_{\omega_\ell} = -\frac{L}{c}\left\{\cos\alpha\left[3\left(\frac{d\alpha}{d\Omega}\right)^2 + 3\Omega\frac{d\alpha}{d\Omega}\frac{d^2\alpha}{d\Omega^2}\right] + \sin\alpha\left[3\frac{d^2\alpha}{d\Omega^2} + \Omega\frac{d^3\alpha}{d\Omega^3} - \Omega\left(\frac{d\alpha}{d\Omega}\right)^3\right]\right\}\Bigg|_{\omega_\ell}$$
$$\approx -\frac{3L}{c}\left[\left(\frac{d\alpha}{d\Omega}\right)^2 + \Omega\frac{d\alpha}{d\Omega}\frac{d^2\alpha}{d\Omega^2}\right]\Bigg|_{\omega_\ell}. \qquad (2.77)$$

where the last expression is a result of $\alpha(\omega_\ell) = 0$.

The most widely used optical devices for angular dispersion are prisms and gratings. To determine the dispersion introduced by them we need to specify not only the quantity $\alpha(\Omega)$ in the expressions derived previously, but also the optical surfaces between which the path is being calculated. Indeed, we have assumed in the previous calculation that the beam started as a plane wave (plane reference surface normal to the initial beam) and terminates in a plane normal to the ray at a reference optical frequency ω_ℓ. The choice of that terminal plane is as arbitrary as that of the reference frequency ω_ℓ (cf. Section 1.1.1). After some propagation distance, the various spectral component of the pulse will have separated, and a finite size detector will only record a portion of the pulse spectrum.

Therefore, the "dispersion" of an element has only meaning in the context of a particular application, which will associate reference surfaces to that element. This is the case when an element is associated with a cavity, as will be considered in the next section. In the following sections, we will consider combinations of elements of which the angular dispersion is compensated. In that case, a natural reference surface is the normal to the beam.

2.5.4. GVD of a Cavity Containing a Single Prism

Dispersion control is an important aspect in the development of fs sources. The most elementary laser cavity as sketched in Figure 2.23 has an element with angular dispersion. The dispersive element could be the Brewster angle

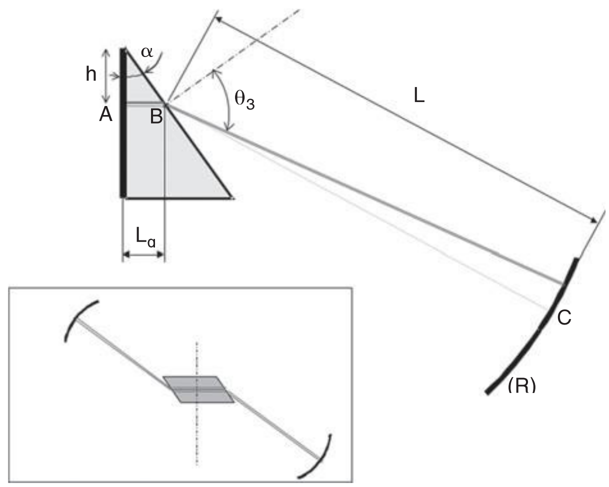

Figure 2.23 Example of a cavity with a single right angle prism. The side of the right angle is an end mirror of the cavity. The cavity is terminated by a curved mirror of radius of curvature R, at a distance L from the Brewster angle exit face of the prism. Stability of the cavity requires that $L + \overline{AB}/n < R$. Translation of the prism allows for an adjustment of the pathlength in glass L_g. The inset shows that this calculation applies to a symmetric cavity with a Brewster angle laser rod and two spherical mirrors.

laser rod itself. The cavity will be typically terminated by a curved mirror. The two reference surfaces to consider are the two end mirrors of the cavity. We have seen that negative GVD is typically associated with angular dispersion, and positive GVD with the propagation through a glass prism or laser rod.[9] One might therefore expect to be able to tune the GVD in the arrangement of Fig. 2.23 from a negative to a positive value. An exact calculation of the frequency dependence presented shows that this is not the case, and that the GVD of this cavity is always positive.

A combination of elements with a tunable positive dispersion can also be desirable in a fs laser cavity. We will consider the case of the linear cavity sketched in Fig. 2.23, whose GVD can be determined analytically.

The cavity is terminated on one end by the plane face of the prism, on the other end by a spherical mirror of curvature R. The prism–mirror distance measured at the central frequency ω_ℓ is L. The beam originates from a distance h from the

[9] It is generally the case—but not always—that optical elements in the visible have positive GVD.

apex of the prism (angle α), such that the pathlength in glass can be written as $L_g = h \tan\alpha$. For the sake of notation simplification, we define:

$$
\begin{aligned}
a &= \frac{h \tan\alpha}{c} \\
b &= \frac{L}{2}\left(1 - \frac{L}{R}\right).
\end{aligned}
\tag{2.78}
$$

The total phase shift for one half cavity roundtrip is $\Psi(\Omega) = \Psi_{AB}(\Omega) + \Psi_{BC}(\Omega)$. The phase shift through the glass here is simply $-k(\Omega)L_g = -\Psi_{AB}(\Omega)$, with $\Psi_{AB}(\Omega)$ given by:

$$
\begin{aligned}
\Psi_{AB}(\Omega) &= \Psi_0 + \left.\frac{d\Psi}{d\Omega}\right|_{\omega_\ell} \Delta\Omega + \frac{1}{2}\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} (\Delta\Omega)^2 + \cdots \\
&\approx \Psi_0 + a\left[\Omega\frac{dn}{d\Omega} + n(\Omega)\right]_{\omega_\ell} \Delta\Omega + \frac{1}{2} a \left[2\frac{dn}{d\Omega} + \Omega\frac{d^2 n}{d\Omega^2}\right]_{\omega_\ell} (\Delta\Omega)^2,
\end{aligned}
\tag{2.79}
$$

where $\Delta\Omega = \Omega - \omega_\ell$. For the path in air, we have a phase shift $-k\overline{BC} = -\Psi_{BC}(\Omega)$, with

$$
\Psi_{BC}(\Omega) = \frac{\Omega}{c}\left[L + \frac{L}{2}\left(1 - \frac{L}{R}\right)\Delta\theta^2\right] = \frac{\Omega}{c}\left[L + b\Delta\theta^2\right], \tag{2.80}
$$

where $\Delta\theta$ is the departure of dispersion angle from the diffraction angle at ω_ℓ. Within the small angle approximation, we have for $\Delta\theta$:

$$
\Delta\theta \approx \Delta\Omega \frac{\sin\alpha}{\cos\theta_3}\frac{dn(\Omega)}{d\Omega} = \Delta\Omega \frac{dn(\Omega)}{d\Omega}. \tag{2.81}
$$

The last equality ($\sin\alpha = \cos\theta_3$) applies to the case where θ_3 equals the Brewster angle. The GVD dispersion of this cavity is thus:

$$
\begin{aligned}
\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} &= \left.\frac{d^2\Psi_{AB}}{d\Omega^2}\right|_{\omega_\ell} + \left.\frac{d^2\Psi_{BC}}{d\Omega^2}\right|_{\omega_\ell} = a\left(2\frac{dn}{d\Omega} + \Omega\frac{d^2}{d\Omega^2}\right)_{\omega_\ell} \\
&\quad + \frac{2b\omega_\ell}{c}\left(\left.\frac{dn}{d\Omega}\right|_{\omega_\ell}\right)^2.
\end{aligned}
\tag{2.82}
$$

or, using the wavelength dependence of the index of refraction, and taking into account that, for the Brewster prism, $\tan\alpha = 1/n(\omega_\ell)$:

$$\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} = \frac{h}{nc}\left(\frac{\lambda}{2\pi c}\right)\left[\lambda^2 \left.\frac{d^2 n}{d\lambda^2}\right|_{\lambda_\ell} + b\frac{\lambda^3}{\pi c^2}\left(\frac{dn}{d\lambda}\right)^2\bigg|_{\lambda_\ell}\right]. \tag{2.83}$$

The stability of the cavity requires that $R > L$ and that the coefficients a and b be positive. In the visible range, most glasses have a positive GVD ($k'' > 0$ or $d^2n/d\lambda^2 > 0$). Therefore, in a cavity with a single prism as sketched in Fig. 2.23, the GVD is adjustable through the parameter h, but always positive.

The calculation applies to a simple solid state laser cavity as sketched in the inset of Fig. 2.23, with a Brewster angle laser rod. The contributions to the dispersion from each side of the dash–dotted line are additive. Even in this simple example, we see that the total dispersion is not only because of the propagation through the glass, but there is also another contribution because of angular dispersion. It is interesting to compare Eq. (2.76), which gives a general formula associated with angular dispersion, with Eq. (2.83). Both expressions involve the square of the angular dispersion, but with opposite sign.

Femtosecond pulses have been obtained through adjustable GVD compensation with a single prism in a dye ring laser cavity [34]. As in the case of Fig. 2.23, the spectral narrowing that would normally take place because of the angular dispersion of the prism was neutralized by having the apex of the prism at a waist of the resonator. In that particular case, the adjustable positive dispersion of the prism provided pulse compression because of the negative chirp introduced by saturable absorption below resonance, as detailed in Chapter 5.

2.5.5. Group Velocity Control with Pairs of Prisms

2.5.5.1. Pairs of Elements

In most applications, a second element will be associated to the first one, such that the angular dispersion introduced by the first element is compensated, and all frequency components of the beam are parallel again, as sketched in Figure 2.24. The elements will generally be prisms or gratings.

As before, we start from a first reference surface A normal to the beam. It seems then meaningful to chose the second reference surface B at the exit of the system that is normal to the beam. There is no longer an ambiguity in the choice of a reference surface, as in the previous section with a single dispersive element. At any particular frequency, Fermat's principle states that the optical paths are equal from a point of the wavefront A to the corresponding point on the

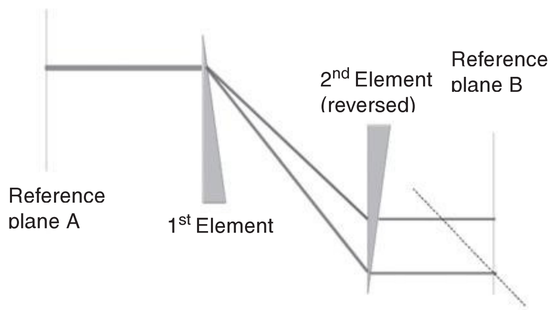

Figure 2.24 Pair of elements with angular dispersion arranged for zero net angular dispersion. The elements are most often prisms or gratings.

wavefront B. This is not to say that these distances are not frequency dependent. The spectral components of the beam are still separated in the transverse direction. For that reason, a pair of prisms or gratings provides a way to "manipulate" the pulse spectrum by spatially filtering (amplitude or phase filter) the various Fourier components.

2.5.5.2. Calculation for Matched Isosceles Prisms

One of the most commonly encountered cases of Fig. 2.24, is that where the two angular dispersive elements are isosceles prisms. Prisms have the advantage of smaller insertion losses, which is particularly important with the low gain solid state lasers used for fs applications. To compensate the angular dispersion, the two prisms are put in opposition, in such a way that, to any face of one prism corresponds a parallel face of the other prism (Figure 2.25).

In this section, we consider only the GVD introduced by the prism sequence. The associated pulse front tilt and the effect of beam divergence will be discussed in Section 2.4 using wave optics. There are numerous contributions to the group velocity dispersion that makes this problem rather complex:

(a) GVD because of propagation in glass for a distance L
(b) GVD introduced by the changes in optical path L in each prism, because of angular dispersion
(c) GVD because of the angular dispersion after one prism, propagation of the beam over a distance ℓ, and as a result propagation through different thicknesses of glass at the next prism.

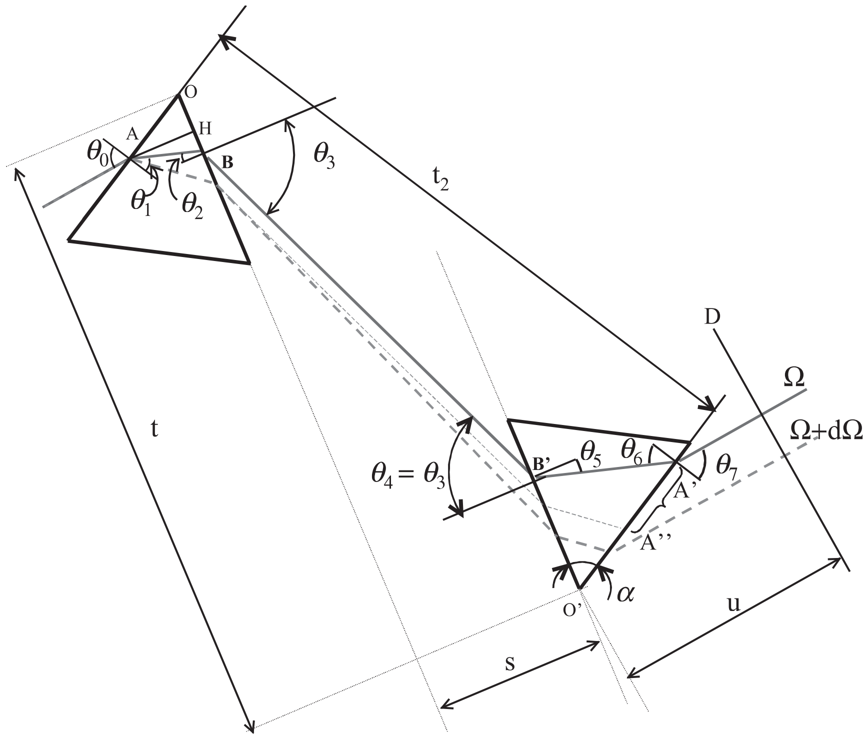

Figure 2.25 Typical two prisms sequence as used in fs laser cavities. The relative position of the prisms is defined by the distance t and the spacing s between the parallel faces OB and $O'B'$. The initial beam enters the prism at a distance $\overline{OA} = a$ from the apex. The distance t_2 between the parallel faces OA and $O'A'$ is $t_2 = t \sin\alpha + s\cos\alpha$. The solid line $ABB'A'D$ traces the beam path at an arbitrary frequency Ω. The beam at the frequency upshifted by $d\Omega$ is represented by the dashed line. The dotted line indicates what the optical path would be in the second prism, if the distance $\overline{BB'}$ were reduced to zero (this situation is detailed in Fig. 2.26). D is a point on the phase front a distance u from the apex O' of the second prism. In most cases we will associate the beam path for a ray at Ω with the path of a ray at the center frequency ω_ℓ.

The optical path $\overline{ABB'A'D}$ at a frequency Ω is represented by the solid line in Fig. 2.25, while the path for a ray upshifted by $d\Omega$ is represented by the dashed line. The successive angles of incidence–refraction are θ_0 and θ_1 at point A, θ_2 and θ_3 at point B, θ_4 and θ_5 at point B', and finally θ_6 and θ_7 at point A'. The two prisms are identical, with equal apex angle α and with pairs of faces oriented parallel as shown in Fig. 2.25. At any wavelength or frequency Ω:

- $\theta_3 = \theta_4$
- $\theta_2 = \theta_5$
- $\theta_1 = \theta_6$
- $\theta_0 = \theta_7$
- $\theta_1 + \theta_2 = \alpha$
- $d\theta_1/d\Omega = -d\theta_2/d\Omega$.

If the prisms are used at minimum deviation at the central wavelength, $\theta_0 = \theta_3 = \theta_4 = \theta_7$. If, in addition to being used at minimum deviation, the prisms are cut for Brewster incidence, the apex angles of both prisms are $\alpha = \pi - 2\theta_0 = \pi - 2\arctan(n)$.

The challenge is to find the frequency dependence of the optical path $\overline{ABB'A'D}$. The initial (geometrical) conditions are defined by

- the distance $a = \overline{OA}$ from the point of impact of the beam to the apex O of the first prisms. For convenience, we will use in the calculations the distance $\overline{OH} = h = \overline{OA}\cos\alpha = a\cos\alpha$,
- the separation s between the parallel faces of the prisms,
- the distance t between the apex O and O', measured along the exit face of the first prism, cf. Fig. 2.25.

The changes in path length because of dispersion can be understood from a glance at the figure, comparing the optical paths at Ω (solid line) and $\Omega + d\Omega$ (dashed line). The contributions that increase the path length are:

1. positive dispersion because of propagation through the prism material of positive dispersion ($\overline{AB}$ and $\overline{B'A'}$)
2. positive dispersion because of the increased path length $\overline{BB'}$ in air (increment $\overline{SB'''}$ in Fig. 2.27)
3. positive dispersion because of the increased path length $\overline{A'D}$ in air (projection of $\overline{A'A'''}$ along the beam propagation direction).

The contributions that decrease the path length (negative dispersion) of the frequency upshifted beam can best be understood with Figures 2.26 and 2.27. Figure 2.26 shows the configurations of the beams if the two prisms were brought together, i.e., $\overline{BB'} = 0$. Figure 2.27 shows an expanded view of the second prism. The negative dispersion contributions emanate from:

1. The shortened path length in glass because of the angular dispersion ($\overline{AA''}$ versus $\overline{AA'}$ in Fig. 2.26).

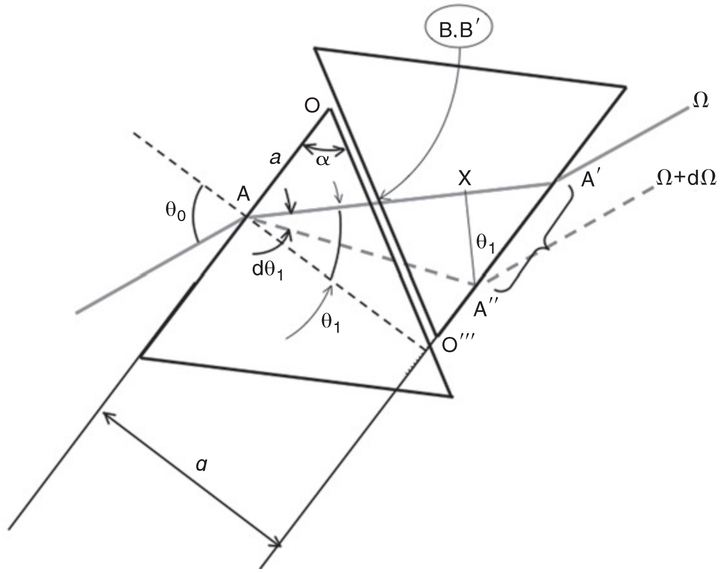

Figure 2.26 Beam passage through the two prisms, when the distance $\overline{BB'}$ (in Fig. 2.25) has been reduced to zero. The distance between the apexes O and O' has been reduced to $\overline{OO'''} = t - \overline{BB'}\sin\theta_3$ (referring to Fig. 2.25). The distance between parallel faces is then $g = \overline{OO'''}\sin\alpha = (t - \overline{BB'}\sin\theta_3)\sin\alpha$.

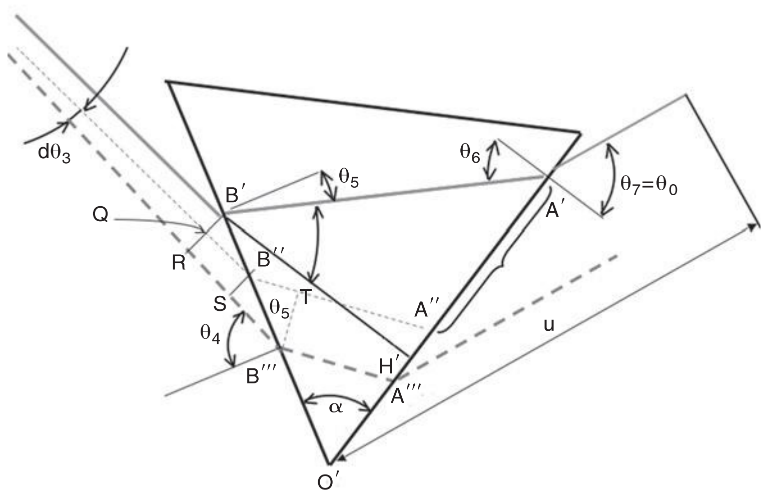

Figure 2.27 Details of the beam passage through the second prism.

2. the shorter path length in the second prism because of the deflection of the beam by the first one (path difference $\overline{B''T}$ in Fig. 2.27).

Path Through Glass

The total path in glass is $L_g = \overline{AB} + \overline{B'A'}$ where $\overline{AB} = a \sin\alpha / \cos\theta_2$ and $\overline{B'A'} = \overline{O'B'} \sin\alpha / \cos(\alpha - \theta_2) = \overline{O'B'} \sin\alpha / \cos\theta_1$, with:

$$\overline{O'B'} = t - s\tan\theta_3 - a(\cos\alpha + \sin\alpha\tan\theta_2). \tag{2.84}$$

We thus have for the total transmitted path in glass:

$$\begin{aligned} L_g = \overline{AB} + \overline{B'A'} &= \frac{a\sin\alpha}{\cos\theta_2} + [t - s\tan\theta_3 - a(\cos\alpha + \sin\alpha\tan\theta_2)]\frac{\sin\alpha}{\cos\theta_1} \\ &= (t - s\tan\theta_3)\frac{\sin\alpha}{\cos\theta_1}. \end{aligned} \tag{2.85}$$

As expected, the total path through glass is independent of the starting position defined by a. If the two prisms are brought together as in Fig. 2.26, they act as a slab of glass with parallel faces, of thickness $g = L_g \cos\theta_1$. There are three contributions to the optical path change from Fig. 2.26: one because of the change in index, a second because of the change in angle, and a third because the path length L_g is frequency dependent. Taking the derivative of $\Omega n L_g / c$ with L_g defined by Eq. (2.85):

$$\begin{aligned} \frac{d(kL_g)}{d\Omega} &= \frac{d}{d\Omega}\left(\frac{n\Omega L_g}{c}\right) \\ &= \frac{L_g}{c}\left(n + \Omega\frac{dn}{d\Omega}\right) + \left(\frac{n\Omega L_g}{c}\tan\theta_1\right)\frac{d\theta_1}{d\Omega} \\ &\quad - \left(\frac{n\Omega s}{c\cos^2\theta_3}\right)\frac{\sin\alpha}{\cos\theta_1}\frac{d\theta_3}{d\Omega} \end{aligned} \tag{2.86}$$

The first term can be attributed solely to material dispersion. The next term is the change in length in glass because of the angular dispersion $d\theta_1/d\Omega$, and the last expresses the reduction in path length in the second prism because of the propagation of the angularly dispersed beam in air. The expression above only partly accounts for the energy tilt associated with the angular dispersion $d\theta_3/d\Omega$. Another contribution arises from the path through air to a reference plane.

Path through Air between and after the Prisms

We have to account for the contributions of the pathlengths $\overline{BB'}$ and $\overline{A'D}$ to the group delay:

$$\frac{d}{d\Omega}\left(\frac{\Omega}{c}\overline{BB'}\right) = \frac{\Omega}{c}\frac{d\overline{BB'}}{d\Omega} + \frac{\overline{BB'}}{c}. \tag{2.87}$$

For the path $\overline{BB'} = s/\cos\theta_3$, there is only a change in length equal to $\overline{SB'''}$, which can be obtained by either differentiating $s/\cos\theta_3$, or simply from geometrical considerations using Fig. 2.25 ($\overline{SB'''} = \overline{SB''}\tan\theta_3 = \overline{BB'}\tan\theta_3 d\theta_3$):

$$\frac{d\overline{BB'}}{d\Omega} = \frac{s}{\cos\theta_3}\tan\theta_3\frac{d\theta_3}{d\Omega}. \tag{2.88}$$

The path in air after the second prism can be expressed as:

$$\overline{A'D} = u - \overline{O'A'}\sin\theta_0. \tag{2.89}$$

Because u is not a function of e, the contribution to the group delay is:

$$\frac{1}{c}\frac{d\left(\Omega\overline{A'D}\right)}{d\Omega} = -\frac{\sin\theta_0}{c}\frac{d}{d\Omega}(\Omega\overline{O'A'}). \tag{2.90}$$

For $\overline{O'A'}$ we find:

$$\begin{aligned}
\overline{O'A'} &= \overline{O'H'} + \overline{H'A'} = \overline{O'B'}\left[\cos\alpha + \sin\alpha\tan(\alpha-\theta_2)\right] \\
&= \left[t - s\tan\theta_3 - a(\cos\alpha + \sin\alpha\tan\theta_2)\right]\left[\cos\alpha + \sin\alpha\tan(\alpha-\theta_2)\right] \\
&= \left[t - s\tan\theta_3 - a\frac{\cos\theta_1}{\cos\theta_2}\right]\left[\cos\alpha + \sin\alpha\tan(\alpha-\theta_2)\right] \\
&= \left[t - s\tan\theta_3\right]\left[\cos\alpha + \sin\alpha\tan\theta_1\right] - a,
\end{aligned} \tag{2.91}$$

where we have used $\cos\alpha + \sin\alpha\tan\theta_2 = \cos(\alpha-\theta_2)\cos\theta_2$. The contribution of $\overline{A'D}$ to the group delay is:

$$\begin{aligned}
-\frac{\sin\theta_0}{c}\frac{d(\Omega\overline{O'A'})}{d\Omega} &= \frac{\overline{O'A'}\sin\theta_0}{c} - \frac{\Omega s\sin\theta_0}{c\cos^2\theta_3}\left[\cos\alpha + \sin\alpha\tan\theta_1\right]\frac{d\theta_3}{d\Omega} \\
&\quad + \frac{\Omega\sin\theta_0}{c}\left[t - s\tan\theta_3\right]\frac{\sin\alpha}{\cos^2\theta_1}\frac{d\theta_1}{d\Omega}
\end{aligned}$$

$$= -\frac{\overline{A'D}}{c} - \frac{n\Omega s}{c\cos^2\theta_3}\left[\cos\alpha\sin\theta_1 + \sin\alpha\frac{\sin^2\theta_1}{\cos\theta_1}\right]\frac{d\theta_3}{d\Omega}$$

$$+ \frac{n\Omega}{c}[t - s\tan\theta_3]\frac{\sin\alpha\sin\theta_1}{\cos^2\theta_1}\frac{d\theta_1}{d\Omega}. \tag{2.92}$$

In the last equation we used the fact that u is an arbitrary constant, for example zero, so that $\overline{A'D} = -\overline{O'A'}\sin\theta_0$.

Total Path in Glass and Air

After adding all contributions to the total phase

$$\Psi = \frac{\Omega}{c}\left(nL_g + \overline{BB'} + \overline{A'D}\right),$$

we obtain for the group delay using Eqs. (2.86), (2.87), (2.88), (2.90), and (2.92):

$$\frac{d\Psi}{d\Omega} = \frac{d}{d\Omega}\left(\frac{\Omega nL_g}{c}\right) + \frac{d}{d\Omega}\left(\frac{\Omega\overline{BB'}}{c}\right) + \frac{d}{d\Omega}\left(\frac{\Omega\overline{A'D}}{c}\right)$$

$$= \frac{nL_g}{c} + \frac{(\overline{BB'} + \overline{A'D})}{c} + \frac{L_g\Omega}{c}\frac{dn}{d\Omega} + \frac{n\Omega L_g}{c}\tan\theta_1\frac{d\theta_1}{d\Omega}$$

$$+ \left\{-\frac{n\Omega s}{c\cos^2\theta_3}\frac{\sin\alpha}{\cos\theta_1} + \frac{\Omega s}{c\cos\theta_3}\tan\theta_3 + \frac{n\Omega s}{c\cos^2\theta_3}\left[\cos\alpha\sin\theta_1\right.\right.$$

$$\left.\left. + \frac{\sin\alpha\sin^2\theta_1}{\cos\theta_1}\right]\right\}\frac{d\theta_3}{d\Omega} - \frac{n\Omega}{c}[t - s\tan\theta_3]\frac{\sin\alpha\sin\theta_1}{\cos^2\theta_1}\frac{d\theta_1}{d\Omega}$$

$$= \frac{OPL(ABB'A'D)}{c} + \frac{L_g\Omega}{c}\frac{dn}{d\Omega}$$

$$+ \left(\frac{\Omega s}{c\cos^2\theta_3}\right)(-n\sin\alpha\cos\theta_1 + \cos\theta_3\tan\theta_3 + n\cos\alpha\sin\theta_1)\frac{d\theta_3}{d\Omega}, \tag{2.93}$$

where we have defined the optical path length $OPL(ABB'A'D) = nL_g + (\overline{BB'} + \overline{A'D})$. The factor preceding $d\theta_3/d\Omega$ cancels, because:

$$\cos\alpha\sin\theta_1 - \sin\alpha\cos\theta_1 + \frac{\sin\theta_3}{n}$$

$$= \cos(\theta_1 + \theta_2)\sin\theta_1 - \sin(\theta_1 + \theta_2)\cos\theta_1 + \sin\theta_2$$

$$= 0.$$

The complete expression for the group delay through the pair of prism reduces to:

$$\boxed{\frac{d\Psi}{d\Omega} = \frac{OPL(ABB'A'D)}{c} + \frac{L_g\Omega}{c}\frac{dn}{d\Omega}.} \tag{2.94}$$

The first terms in the last equation represents the travel delay at the phase velocity:

$$\frac{OPL(ABB'A'D)}{c} = \frac{L_g n}{c} + \frac{s}{c\cos\theta_3} + \frac{\overline{A'D}}{c}. \tag{2.95}$$

The second part of Eq. (2.94) is the carrier to envelope delay caused by the pair of prisms[10]:

$$\tau_{CE}(\Omega) = \frac{\Omega}{c}\frac{d}{d\Omega}OPL(ABB'A'D) = \frac{L_g\Omega}{c}\frac{dn}{d\Omega}. \tag{2.96}$$

The second derivative of the phase, obtained by taking the derivative of Eq. (2.94), is:

$$\begin{aligned}
\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} &= L_g\left[2\left.\frac{dn}{d\Omega}\right|_{\omega_\ell} + \omega_\ell\left.\frac{d^2n}{d\Omega^2}\right|_{\omega_\ell}\right] \\
&\quad - \frac{\omega_\ell}{c\cos\theta_1}\left.\frac{dn}{d\Omega}\right|_{\omega_\ell}\left[\frac{s\sin\alpha}{\cos^2\theta_3}\left.\frac{d\theta_3}{d\Omega}\right|_{\omega_\ell}\right] \\
&\quad + \frac{\omega_\ell}{c}\left.\frac{dn}{d\Omega}\right|_{\omega_\ell}\left[L_g\tan\theta_1\left.\frac{d\theta_1}{d\Omega}\right|_{\omega_\ell}\right].
\end{aligned} \tag{2.97}$$

The derivatives with respect to Ω are related. By differentiating Snell's law for the first interface:

$$d\theta_1 = -\frac{\tan\theta_1}{n}dn = -d\theta_2. \tag{2.98}$$

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[10]We are assuming that the prisms are in vacuum, i.e., the contribution to the dispersion from air is neglected.

For the second interface, taking the previous relation into account, we find:

$$\cos\theta_3 d\theta_3 = n\cos\theta_2 d\theta_2 + \sin\theta_2 dn = n\cos\theta_2 \left(\frac{\tan\theta_1}{n} + \sin\theta_2\right) dn$$

$$= (\cos\theta_2 \tan\theta_1 + \sin\theta_2)\, dn = \frac{\sin\alpha}{\cos\theta_1} dn. \tag{2.99}$$

or:

$$d\theta_3 = \frac{\sin\alpha}{\cos\theta_1 \cos\theta_3} dn. \tag{2.100}$$

Therefore, the second-order dispersion Eq. (2.97) reduces to an easily interpretable form:

$$\boxed{\begin{aligned} \left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} &= \frac{L_g}{c}\left[2\left.\frac{dn}{d\Omega}\right|_{\omega_\ell} + \omega_\ell \left.\frac{d^2n}{d\Omega^2}\right|_{\omega_\ell}\right] \\ &\quad - \frac{\omega_\ell}{c}\left(\frac{s}{\cos\theta_3}\right)\left(\left.\frac{d\theta_3}{d\Omega}\right|_{\omega_\ell}\right)^2 - \frac{n\omega_\ell}{c} L_g \left(\left.\frac{d\theta_1}{d\Omega}\right|_{\omega_\ell}\right)^2 \end{aligned}} \tag{2.101}$$

This equation applies to any pair of identical isosceles prisms in the parallel face configuration represented in Fig. 2.25, for an arbitrary angle of incidence. The GVD is simply the sum of three contributions:

1. The (positive) GVD because of the propagation of the pulse through a thickness of glass L_g.
2. The negative GVD contribution because of the angular dispersion $d\theta_3/d\Omega$ applied to Eq. (2.76) over a distance $\overline{BB'} = s/\cos\theta_3$.
3. The negative GVD contribution because of the angular dispersion $d\theta_1/d\Omega$ (deflection of the beam at the first interface) applied to Eq. (2.76) over a distance L_g in the glass of index n.

In most practical situations it is desirable to write Eq. (2.101) in terms of the input angle of incidence θ_0 and the prism apex angle α. The necessary equations can be derived from Snell's law and Eq. (2.76):

$$\frac{d}{d\Omega}\theta_1 = \left[n^2 - \sin^2(\theta_0)\right]^{-\frac{1}{2}}\left[n\cos\theta_0 \frac{d\theta_0}{d\Omega} - \sin\theta_0 \frac{dn}{d\Omega}\right]$$

$$\frac{d}{d\Omega}\theta_3 = \left[1 - n^2 \sin^2(\alpha - \theta_1)\right]^{-\frac{1}{2}} \left[n \cos(\alpha - \theta_1)\frac{d\theta_1}{d\Omega} + \sin(\alpha - \theta_1)\frac{dn}{d\Omega}\right]. \tag{2.102}$$

where $\theta_1 = \arcsin(n^{-1}\sin\theta_0)$ and $d\theta_0/d\Omega = 0$.

For the particular case of Brewster angle prisms and minimum deviation (symmetric beam path through the prism for $\Omega = \omega_\ell$), we can make the substitutions $d\theta_1/dn = -1/n^2$, and $d\theta_3/dn = 2$. Using $\theta_0 = \theta_3 = \theta_4 = \theta_7$, the various angles are related by:

$$\begin{aligned}
\tan\theta_0 &= n \\
\sin\theta_0 = \cos\theta_1 &= \frac{n}{\sqrt{1+n^2}} \\
\cos\theta_0 = \sin\theta_1 &= \frac{1}{\sqrt{1+n^2}} \\
\sin\alpha &= \frac{2n}{n^2+1}.
\end{aligned} \tag{2.103}$$

The total second-order dispersion in this case becomes:

$$\boxed{\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} = \frac{L_g}{c}\left[2\left.\frac{dn}{d\Omega}\right|_{\omega_\ell} + \omega_\ell \left.\frac{d^2n}{d\Omega^2}\right|_{\omega_\ell}\right] - \frac{\omega_\ell}{c}\left(4L + \frac{L_g}{n^3}\right)\left(\left.\frac{dn}{d\Omega}\right|_{\omega_\ell}\right)^2.} \tag{2.104}$$

where we have introduced the distance between the two prisms measured along the central wavelength $L = s/\cos\theta_3$. In terms of wavelength:

$$\boxed{\left.\frac{d^2\Psi}{d\Omega^2}\right|_{\omega_\ell} = \frac{\lambda_\ell^3}{2\pi c^2}\left[L_g \left.\frac{d^2n}{d\lambda^2}\right|_{\lambda_\ell} - \left(4L + \frac{L_g}{n^3}\right)\left(\left.\frac{dn}{d\lambda}\right|_{\lambda_\ell}\right)^2\right].} \tag{2.105}$$

In many practical devices, $L \gg L_g$ and the second term of Eq. (2.105) reduces to $L(dn/d\lambda)^2$.

It is left as a problem at the end of this chapter to calculate the exact third-order dispersion for a pair of prisms. If the angular dispersion in the glass can be neglected ($L \gg L_g$), the third-order dispersion for a Brewster angle prism is:

$$\Psi'''_{\text{tot}}(\omega_\ell) \approx$$

$$\frac{\lambda_\ell^4}{(2\pi c)^2 c}\left[12L\left(n'^2\left[1 - \lambda_\ell n'(n^{-3} - 2n)\right] + \lambda_\ell n' n''\right) - L_g(3n'' + \lambda_\ell n''')\right]. \tag{2.106}$$

To simplify the notation, we have introduced n', n'' and n''' for the derivatives of n with respect to λ taken at λ_ℓ.

The presence of a negative contribution to the GVD because of angular dispersion offers the possibility of tuning the GVD by changing $L_g = g/\sin\theta_0$ (g is the thickness of the glass slab formed by bringing the two prisms together, as shown in Fig. 2.26). A convenient method is to simply translate one of the prisms perpendicularly to its base, which alters the glass path while keeping the beam deflection constant. It will generally be desirable to avoid a transverse displacement of spectral components at the output of the dispersive device. Two popular prism arrangements which do not separate the spectral components of the pulse are sketched in Figure 2.28. The beam is either sent through two prisms, and retro-reflected by a plane mirror, or sent directly through a sequence of four prisms. In these cases the dispersion as described by Eq. (2.101) doubles. The values of Ψ'', Ψ''', etc. that are best suited to a particular experimental situation can be predetermined through a selection of the optimum prism separation $s/\cos\theta_3$, the glass pathlength L_g, and the material (cf. Table 2.1). Such optimization methods are particularly important for the generation of sub-20 fs pulses in lasers [35,36] that use prisms for GVD control.

In this section we have derived analytical expressions for dispersion terms of increasing order, in the case of identical isoceles prism pairs, in exactly antiparallel configuration. It is also possible by methods of pulse tracing through

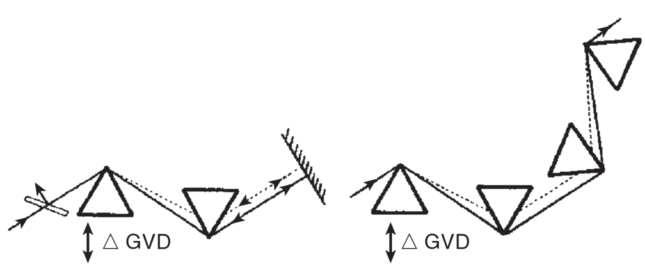

Figure 2.28 Setups for adjustable GVD without transverse displacement of spectral components.

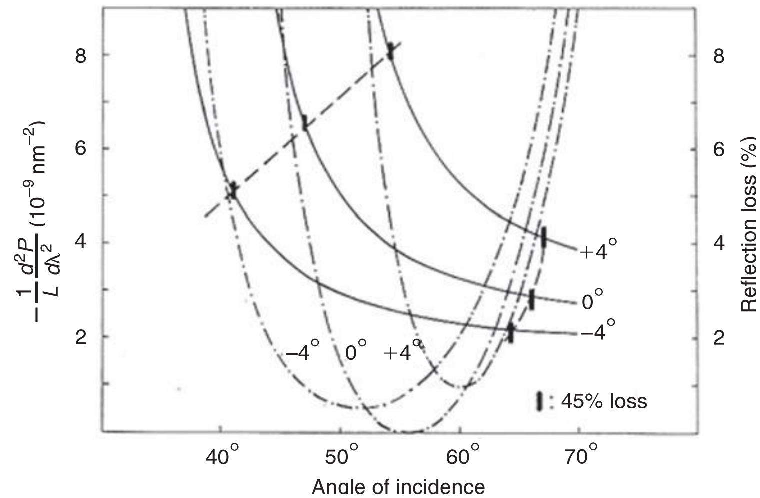

Figure 2.29 Dispersion (solid lines) and reflection losses (dash–dotted lines) of a two-prism sequence (SQ1—fused silica) as a function of the angle of incidence on the first prism surface. Symmetric beam path through the prism at the central wavelength is assumed. Curves for three different apex angles ($-4°$, $0°$, $4°$) relative to $\alpha = 68.9°$ (apex angle for a Brewster prism at 620 nm) are shown. The tic marks on the dashed lines indicate the angle of incidence and the dispersion where the reflection loss is 4.5%. (Adapted from Petrov *et al.* [31]).

the prisms to determine the phase factor at any frequency and angle of incidence [30, 31, 37–39]. The more complex studies revealed that the GVD and the transmission factor R [as defined in Eq. (2.71)] depend on the angle of incidence and apex angle of the prism. In addition, any deviation from the Brewster condition increases the reflection losses. An example is shown in Fig. 2.29.

2.5.6. GVD Introduced by Gratings

Gratings can produce larger angular dispersion than prisms. The resulting negative GVD was first utilized by Treacy [28] to compress pulses of a Nd:glass laser. In complete analogy with prisms, the simplest practical device consists of two identical elements arranged as in Figure 2.30 for zero net angular dispersion. The dispersion introduced by a pair of parallel gratings can be determined by tracing the frequency dependent ray path. The optical path length $\overline{ACP}$ between A and an output wavefront $\overline{PP_0}$ is frequency dependent and can be determined

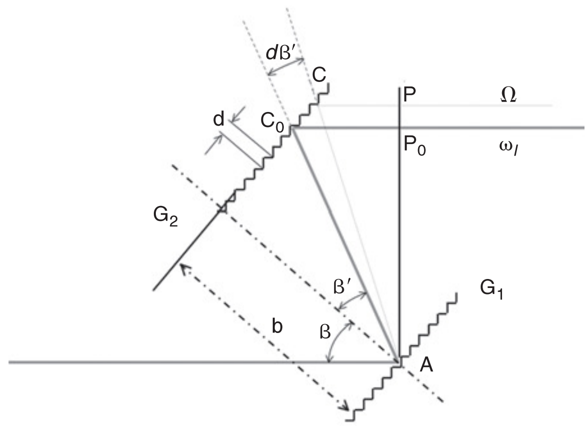

Figure 2.30 Two parallel gratings produce GVD without net angular dispersion. For convenience a reference wavefront is assumed so that the extension of $\overline{PP_0}$ intersects G_1 at A.

with help of Fig. (2.30) to be:

$$\overline{ACP} = \frac{b}{\cos(\beta')}\left[1 + \cos(\beta' + \beta)\right] \tag{2.107}$$

where β is the angle of incidence, β' is the diffraction angle for the frequency component Ω and b is the normal separation between G_1 and G_2. If we restrict our consideration to first-order diffraction, the angle of incidence and the diffraction angle are related through the grating equation

$$\sin\beta' - \sin\beta = \frac{2\pi c}{\Omega d} \tag{2.108}$$

where d is the grating constant. The situation with gratings is however different than with prisms, in the sense that the optical path of two parallel rays out of grating G_1 impinging on adjacent grooves of grating G_2 will see an optical path difference $\overline{CP} - \overline{C_0P_0}$ of $m\lambda$, m being the diffraction order. Thus, as the angle β' changes with wavelength, the phase factor $\Omega\overline{ACP}/c$ increments by $2m\pi$ each time the ray $\overline{AC}$ passes a period of the ruling of G_2 [28]. Because only the relative phase shift across $\overline{PP_0}$ matters, we may simply count the rulings from the (virtual) intersection of the normal in A with G_2. Thus, for first-order diffraction ($m = 1$),