

Strong coupling constant to four loops in the analytic approach to QCD

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Abstract

The QCD analytic running coupling α_{an} which has no nonphysical singularities for all $Q^2 > 0$ is considered for the initial perturbation theory approximations up to four loop order. The finiteness of the analytic coupling at zero is shown to be a consequence of the asymptotic freedom property of the initial theory. The nonperturbative contributions to the analytic coupling are extracted explicitly. For all $Q > \Lambda$ they are represented in the form of an expansion in inverse powers of Euclidean momentum squared. The effective method for a precise calculation of the analytic running coupling is developed on the basis of the stated expansion. The energy scale evolution of the analytic running coupling for the one- to four-loop cases is studied and the higher loop stability and low dependence on the quark threshold matching conditions in comparison with the perturbative running coupling were found. Normalizing the analytic running coupling at the scale of the rest mass of the Z boson with the world average value of the strong coupling constant, $\alpha_{an}(M_Z^2) = 0.1181 \pm 0.002$, one obtains as a result of the energy scale evolution of the analytic running coupling $\alpha_{an}(M_\tau^2) = 0.2943^{+0.0111}_{-0.0106}$ that is notably lower than the estimations of the coupling strength available at the scale of the mass of the τ lepton.

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I. INTRODUCTION

The strong coupling constant α_s is the basic parameter of Quantum Chromodynamics (QCD) and its determination appears to be one of the most important problems [1, 2, 3]. The perturbation theory supplemented with the renormalization group method works effectively beyond the infrared region. The nonphysical singularities of the perturbation theory arise in the infrared region of QCD and should be canceled by the nonperturbative contributions. The nonperturbative contributions arise quite naturally in an analytic approach to QCD (for a review see, e.g. [4]). The idea of the procedure goes back to Refs. [5, 6] devoted to the ghost pole problem in QFT. The foundation of the procedure is the principle of summation of imaginary parts of the perturbation theory terms. Then, the Källen – Lehmann spectral representation results in the expressions without nonphysical singularities. In recent papers [7, 8] it is suggested to solve the ghost pole problem in QCD demanding the $\alpha_s(Q^2)$ be analytic in Q^2 (to compare with the dispersive approach [9]). As a result, instead of the one-loop expression $\alpha_s^{(1)}(Q^2) = (4\pi/b_0)/\ln(Q^2/\Lambda^2)$ taking into account the leading logarithms and having the ghost pole at $Q^2 = \Lambda^2$ (Q^2 is the Euclidean momentum squared), one obtains the expression

$$\alpha_{an}^{(1)}(Q^2) = \frac{4\pi}{b_0} \left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right]. \quad (1)$$

Eq. (1) is an analytic function in the complex Q^2 -plane with a cut along the negative real semiaxis. The pole of the perturbative running coupling at $Q^2 = \Lambda^2$ is canceled by the nonperturbative contribution [$\Lambda^2 \simeq \mu^2 \exp\{-4\pi/(b_0\alpha_{an}(\mu^2))\}$ at $\alpha_{an}(\mu^2) \rightarrow 0$] and the value $\alpha_{an}^{(1)}(0) = 4\pi/b_0$ appeared finite and independent of Λ . The important feature of the "analyticization procedure" discovered [7, 8] is the stability property of the value of the "analytically improved" running coupling constant at zero with respect to higher order corrections, $\alpha_{an}^{(1)}(0) = \alpha_{an}^{(2)}(0) = \alpha_{an}^{(3)}(0)$. Though the derivative of the analytic running coupling is infinite at zero, $\alpha_{an}(Q^2)$ turns out to be stable with respect to higher order corrections in the infrared region as a whole.

The 1-loop order nonperturbative contribution in Eq. (1) can be presented as convergent at $Q^2 > \Lambda^2$ of constant signs series in the inverse powers of the momentum squared. For a "standard" as well as for iterative 2-loop perturbative input the nonperturbative contributions in analytic running coupling are calculated explicitly in Ref. [10]. In the ultraviolet region the nonperturbative contributions can also be represented as a series in inverse powers of the momentum squared with different coefficients of the expansion. The nonperturbative contributions to $\alpha_{an}(Q^2)$ up to 3-loop order in analytic approach to QCD are studied in Refs. [11], [12]. To handle the singularities originating from the perturbative input the method which is more general than that of Ref. [10] was developed. In Ref. [12] the momentum dependence of α_{an} and its perturbative and nonperturbative components in the infrared region are analysed. For the standard perturbative input the higher loop stability and low sensitivity with respect to the c quark threshold matching conditions were found for α_{an} . In Ref. [13] the nonperturbative contributions for the 4-loop case are considered briefly.

In this paper the momentum dependence of the analytic running coupling up to 4-loop order is studied. In parallel, the behavior of the perturbative component is given

for convenience of comparison. In Section 2 we generalize slightly the standard four-loop solution for α_s , find and study the spectral density for α_{an} and then we prove the important property $\alpha_{an}(0) = 4\pi/b_0$. In Section 3 we extract from α_{an} the initial perturbative contribution α^{pt} and find in an explicit form the nonperturbative contribution α_{an}^{npt} . We develop a technique of integration in the vicinity of severe singularities of the perturbation theory in the infrared region and represent α_{an}^{npt} in the form of a finite limits integral. In Section 4 the power series representation for α_{an}^{npt} at $Q > \Lambda$ is obtained. In Section 5 we study the momentum behavior of α_{an} . We consider it instructive to normalize α_{an} and α^{pt} at M_Z and then compare their behavior to estimate the nonperturbative contributions at all momenta. We consider two methods of matching of the solutions with different numbers n_f of active quark flavors. Finally in Section 6 we give our conclusions. In the Appendix we give the explicit formulas which allow one to simplify the integration in the vicinity of the singularities of the standard perturbation theory input.

II. FROM THE RUNNING COUPLING TO THE ANALYTIC RUNNING COUPLING

The behavior of the QCD running coupling $\alpha_s(Q^2)$ is defined by the renormalization group equation

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \beta_2 \alpha_s^4 + \beta_3 \alpha_s^5 + O(\alpha_s^6), \quad (2)$$

where the coefficients [14] — [17]

$$\begin{aligned} \beta_0 &= -\frac{1}{4\pi} b_0, \quad b_0 = 11 - \frac{2}{3} n_f, \\ \beta_1 &= -\frac{1}{8\pi^2} b_1, \quad b_1 = 51 - \frac{19}{3} n_f, \\ \beta_2 &= -\frac{1}{128\pi^3} b_2, \quad b_2 = 2857 - \frac{5033}{9} n_f + \frac{325}{27} n_f^2, \\ \beta_3 &= -\frac{1}{256\pi^4} b_3, \quad b_3 = \frac{149753}{6} + 3564\zeta_3 \\ &\quad - \left(\frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f + \left(\frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3. \end{aligned} \quad (3)$$

Here n_f is the number of active quark flavors and ζ is the Riemann zeta-function, $\zeta_3 = \zeta(3) = 1.202056903\dots$. The first two coefficients β_0, β_1 do not depend on the renormalization scheme choice. The next coefficients do depend on it. Calculated within the \overline{MS} -scheme in an arbitrary covariant gauge for the gluon field they appeared to be independent of the gauge parameter choice. Values of the coefficients (3) are given in Table 1. All these coefficients are small enough and decrease in absolute value with n_f increasing. All the coefficients are negative except β_2 at $n_f = 6$.

The integration of Eq. (2) yields

$$\begin{aligned} \frac{1}{\alpha_s(Q^2)} &+ \frac{\beta_1}{\beta_0} \ln \alpha_s(Q^2) + \frac{1}{\beta_0^2} (\beta_0 \beta_2 - \beta_1^2) \alpha_s(Q^2) + \frac{1}{2\beta_0^3} (\beta_1^3 - 2\beta_0 \beta_1 \beta_2 + \beta_0^2 \beta_3) \alpha_s^2(Q^2) \\ &+ O(\alpha_s^3(Q^2)) = -\beta_0 \ln(Q^2/\Lambda^2) + \bar{C}. \end{aligned} \quad (4)$$

Table 1: n_f dependence of \overline{MS} values of β_i ($i = \overline{0, 3}$), b , κ , $\bar{\kappa}$.

n_f	β_0	β_1	β_2	β_3	b	κ	$\bar{\kappa}$
0	-0.87535	-0.64592	-0.71986	-1.17269	0.84298	0.51033	-1.16716
1	-0.82230	-0.56571	-0.58199	-0.91043	0.83663	0.49541	-1.20019
2	-0.76925	-0.48550	-0.45019	-0.68103	0.82045	0.46922	-1.26081
3	-0.71620	-0.40528	-0.32445	-0.48484	0.79012	0.41467	-1.36791
4	-0.66315	-0.32507	-0.20477	-0.32222	0.73920	0.28506	-1.56255
5	-0.61009	-0.24486	-0.09116	-0.19354	0.65784	-0.07234	-1.95343
6	-0.55704	-0.16465	0.01638	-0.09914	0.53061	-1.33654	-2.94623

The integration constant is represented here as a combination of two constants Λ and \bar{C} . Dimensional constant Λ is a parameter which defines the scale of Q and is used for developing the iteration procedure. Iteratively solving Eq. (4) for $\alpha_s(Q^2)$ at $L = \ln(Q^2/\Lambda^2) \rightarrow \infty$ we obtain

$$\frac{1}{\alpha_s(Q^2)} = -\beta_0 L + \frac{\beta_1}{\beta_0} (\ln L + C) - \frac{\beta_1^2}{\beta_0^3 L} \left(\ln L + C + 1 - \frac{\beta_0 \beta_2}{\beta_1^2} \right) - \frac{\beta_1^3}{2\beta_0^5 L^2} \left[(\ln L + C)^2 - \frac{2\beta_0 \beta_2}{\beta_1^2} (\ln L + C) - 1 + \frac{\beta_0^2 \beta_3}{\beta_1^3} \right] + O\left(\frac{1}{L^3}\right), \quad (5)$$

where $C = \ln(-\beta_0) + (\beta_0/\beta_1)\bar{C}$. Inverting Eq. (5) one obtains

$$\alpha_s(Q^2) = -\frac{1}{\beta_0 L} \left\{ 1 + \frac{\beta_1}{\beta_0^2 L} (\ln L + C) + \frac{\beta_1^2}{\beta_0^4 L^2} \left[(\ln L + C)^2 - (\ln L + C) - 1 + \frac{\beta_0 \beta_2}{\beta_1^2} \right] + \frac{\beta_1^3}{\beta_0^6 L^3} \left[(\ln L + C)^3 - \frac{5}{2} (\ln L + C)^2 - \left(2 - \frac{3\beta_0 \beta_2}{\beta_1^2} \right) (\ln L + C) + \frac{1}{2} - \frac{\beta_0^2 \beta_3}{2\beta_1^3} \right] + O\left(\frac{1}{L^4}\right) \right\}. \quad (6)$$

Within the conventional definition of Λ as $\Lambda_{\overline{MS}}$ [18] one chooses $C = 0$. At that the functional form of the approximate solution for $\alpha_s(Q^2)$ turns out to be somewhat simpler, but it requires distinct $\Lambda_{\overline{MS}}$ for different n_f . With this choice, Eq. (6) at the three loop level corresponds to the standard solution written in the form of the expansion in inverse powers of logarithms [1], and at the four loop level it corresponds to [19]. We shall deal with nonzero C since this freedom can be useful for an optimization of the finite order perturbation calculations. Moreover, in the presence of the n_f -dependent constant C it is possible to construct matched solution of Eq. (2) with universal n_f independent constant Λ [20].

Let us introduce the function $a(x) = (b_0/4\pi)\alpha_s(Q^2)$, where $x = Q^2/\Lambda^2$. Then instead of (6) one can write

$$a(x) = \frac{1}{\ln x} - b \frac{\ln(\ln x) + C}{\ln^2 x} + b^2 \left[\frac{(\ln(\ln x) + C)^2}{\ln^3 x} - \frac{\ln(\ln x) + C}{\ln^3 x} + \frac{\kappa}{\ln^3 x} \right] - b^3 \left[\frac{(\ln(\ln x) + C)^3}{\ln^4 x} - \frac{5}{2} \frac{(\ln(\ln x) + C)^2}{\ln^4 x} + (3\kappa + 1) \frac{\ln(\ln x) + C}{\ln^4 x} + \frac{\bar{\kappa}}{\ln^4 x} \right]. \quad (7)$$

where the coefficients b , κ , and $\bar{\kappa}$ are equal to

$$\begin{aligned} b &= -\frac{\beta_1}{\beta_0^2} = \frac{2b_1}{b_0^2}, \\ \kappa &= -1 + \frac{\beta_0\beta_2}{\beta_1^2} = -1 + \frac{b_0b_2}{8b_1^2}, \\ \bar{\kappa} &= \frac{1}{2} - \frac{\beta_0^2\beta_3}{2\beta_1^3} = \frac{1}{2} - \frac{b_0^2b_3}{16b_1^3}. \end{aligned} \quad (8)$$

The values of parameters b , κ , and $\bar{\kappa}$ of Eq. (7) for different n_f are given in Table 1. At $x \simeq 1$ the perturbative running coupling is singular. At large x the 1-loop term of Eq. (7) defines the ultraviolet behavior of $a(x)$. However, for small x the behavior of the running coupling depends on the approximation we adopt and at $x = 1$ there are singularities of a different analytical structure. Namely, at $x \simeq 1$ the leading singularities are

$$\begin{aligned} a^{(1)}(x) &\simeq \frac{1}{x-1}, \quad a^{(2)}(x) \simeq -\frac{b}{(x-1)^2} \ln(x-1), \\ a^{(3)}(x) &\simeq \frac{b^2}{(x-1)^3} \ln^2(x-1), \quad a^{(4)}(x) \simeq -\frac{b^3}{(x-1)^4} \ln^3(x-1). \end{aligned} \quad (9)$$

From Eqs. (9) we know the leading behavior at $x \simeq 1$ of the additional terms which should cancel the perturbative singularities. But in principle it gives no information on their behavior at large x . The analytic approach removes all these nonphysical singularities in a regular way.

The analytic running coupling is obtained by the integral representation

$$a_{an}(x) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{x+\sigma} \rho(\sigma), \quad (10)$$

where the spectral density $\rho(\sigma) = \text{Im}a_{an}(-\sigma - i0)$. According to the analytic approach to QCD we adopt that $\text{Im}a_{an}(-\sigma - i0) = \text{Im}a(-\sigma - i0)$, where $a(x)$ is the perturbative running coupling. It is clear that dispersively-modified coupling of form (10) has analytical structure which is consistent with causality.

By making the analytic continuation of Eq. (7) into the Minkowski space $x = -\sigma - i0$, one obtains

$$\begin{aligned} a(-\sigma - i0) &= \frac{1}{\ln \sigma - i\pi} - b \frac{\ln(\ln \sigma - i\pi) + C}{(\ln \sigma - i\pi)^2} + b^2 \left\{ \frac{[\ln(\ln \sigma - i\pi) + C]^2}{(\ln \sigma - i\pi)^3} \right. \\ &- \frac{\ln(\ln \sigma - i\pi) + C}{(\ln \sigma - i\pi)^3} + \frac{\kappa}{(\ln \sigma - i\pi)^3} \left. \right\} - b^3 \left\{ \frac{[\ln(\ln \sigma - i\pi) + C]^3}{(\ln \sigma - i\pi)^4} - \frac{5}{2} \frac{[\ln(\ln \sigma - i\pi) + C]^2}{(\ln \sigma - i\pi)^4} \right. \\ &\left. + (3\kappa + 1) \frac{\ln(\ln \sigma - i\pi) + C}{(\ln \sigma - i\pi)^4} + \frac{\bar{\kappa}}{(\ln \sigma - i\pi)^4} \right\}. \end{aligned} \quad (11)$$

Taking an imaginary part of Eq. (11) we find the spectral density

$$\rho(\sigma) = \rho^{(1)}(\sigma) + \Delta\rho^{(2)}(\sigma) + \Delta\rho^{(3)}(\sigma) + \Delta\rho^{(4)}(\sigma), \quad (12)$$

where

$$\rho^{(1)}(\sigma) = \frac{\pi}{t^2 + \pi^2}, \quad (13)$$

$$\Delta\rho^{(2)}(\sigma) = -\frac{b}{(t^2 + \pi^2)^2} \left[2\pi t F_1(t) - (t^2 - \pi^2) F_2(t) \right], \quad (14)$$

$$\begin{aligned} \Delta\rho^{(3)}(\sigma) = & \frac{b^2}{(t^2 + \pi^2)^3} \left[\pi (3t^2 - \pi^2) (F_1^2(t) - F_2^2(t)) - 2t (t^2 - 3\pi^2) F_1(t) F_2(t) \right. \\ & \left. - \pi (3t^2 - \pi^2) F_1(t) + t (t^2 - 3\pi^2) F_2(t) + \pi\kappa (3t^2 - \pi^2) \right], \end{aligned} \quad (15)$$

$$\begin{aligned} \Delta\rho^{(4)}(\sigma) = & -\frac{b^3}{(t^2 + \pi^2)^4} \left[(t^4 - 6\pi^2 t^2 + \pi^4) (F_2^3(t) - 3F_1^2(t) F_2(t)) + 4\pi t (t^2 - \pi^2) (F_1^3(t) \right. \\ & \left. - 3F_1(t) F_2^2(t)) - 10\pi t (t^2 - \pi^2) (F_1^2(t) - F_2^2(t)) + 5 (t^4 - 6\pi^2 t^2 + \pi^4) F_1(t) F_2(t) \right. \\ & \left. + 4\pi (1 + 3\kappa) t (t^2 - \pi^2) F_1(t) - (1 + 3\kappa) (t^4 - 6\pi^2 t^2 + \pi^4) F_2(t) + 4\pi\bar{\kappa} t (t^2 - \pi^2) \right]. \end{aligned} \quad (16)$$

Here $t = \ln(\sigma)$,

$$F_1(t) \equiv \frac{1}{2} \ln(t^2 + \pi^2) + C, \quad F_2(t) \equiv \arccos \frac{t}{\sqrt{t^2 + \pi^2}}, \quad (17)$$

$\rho^{(1)}(\sigma)$ is the 1-loop spectral density and $\Delta\rho^{(l)}(\sigma)$ are higher loop corrections to the spectral density. With Eqs. (10), (12) — (17) the analytic running coupling can be studied, e.g. by numerical methods. For the 1 — 4-loop cases the spectral density of the analytic running coupling is shown in Fig. 1. For the curves in Fig. 1 and in the next Fig. 2 the parameter values $C = 0$, $n_f = 3$ are chosen. Beyond the 1-loop approximation one can see the higher loop stabilization of the spectral density and in the region of $|t| > 10$ it is practically the same for the 2 — 4-loop cases. Integrating the spectral density numerically with replacement of the infinite limits by finite cut parameter T leads to the relative error $\sim 1/T$, and at large T it is important not to lose the higher loop contributions. In Fig. 2 the higher loop corrections to the spectral density are shown. In fact we deal with rapidly oscillating functions and one needs special methods for a precise integration. E.g., for the 4-loop case it is difficult to get a 2-percent accuracy for $\alpha_{an}(M_\tau^2)$ using the standard integration program DGAUSS.

We shall obtain another more effective method for precise calculation of $\alpha_{an}(Q^2)$ which is not connected with the numerical integration.

Function $a(x)$ in Eq. (7) is regular and real for real $x > 1$. Thus, to find the spectral density $\rho(\sigma)$ we can use Schwarz reflection principle $(a(x))^* = a(x^*)$ where x is considered as a complex variable. Then

$$\rho(\sigma) = \frac{1}{2i} (a(-\sigma - i0) - a(-\sigma + i0)). \quad (18)$$

Let us introduce function $\Phi(z)$ of the form

$$\begin{aligned} \Phi(z) = & \frac{1}{z} - b \frac{\ln(z) + C}{z^2} + b^2 \left[\frac{(\ln(z) + C)^2}{z^3} - \frac{\ln(z) + C}{z^3} + \frac{\kappa}{z^3} \right] \\ & - b^3 \left[\frac{(\ln(z) + C)^3}{z^4} - \frac{5}{2} \frac{(\ln(z) + C)^2}{z^4} + (3\kappa + 1) \frac{\ln(z) + C}{z^4} + \frac{\bar{\kappa}}{z^4} \right]. \end{aligned} \quad (19)$$

To choose the main branch of the multivalued function (19) we cut the complex z -plane along the negative semiaxis. Then solution (7) can be written as $a(x) = \Phi(\ln x)$. Function $a(x)$ is unambiguously defined in the complex x -plane with two cuts along the real axis, physical cut from minus infinity to zero and nonphysical one from zero to unity. Then

$$\rho(\sigma) = \frac{1}{2i} (\Phi(\ln \sigma - i\pi) - \Phi(\ln \sigma + i\pi)). \quad (20)$$

By the change of variable of the form $\sigma = \exp(t)$, the analytical expression is derived from (10), (20) as follows:

$$a_{an}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^t}{x + e^t} \times \{\Phi(t - i\pi) - \Phi(t + i\pi)\}. \quad (21)$$

Let us prove that $a_{an}(0) = 1$. It follows from Eq. (21) that

$$\begin{aligned} a_{an}(0) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \{\Phi(t - i\pi) - \Phi(t + i\pi)\} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left\{ \left[\Phi(t - i\pi) - \frac{1}{t - i\pi} \right] - \left[\Phi(t + i\pi) - \frac{1}{t + i\pi} \right] + \left[\frac{1}{t - i\pi} - \frac{1}{t + i\pi} \right] \right\}. \end{aligned} \quad (22)$$

For the first term in Eq. (22) we close the integration contour in the lower half-plane of the complex variable t by the arch of the "infinite" radius without affecting the value of the integral. We can do it because the integrand multiplied by t goes to zero at $|t| \rightarrow \infty$. There are no singularities inside the contour, and thus we obtain a zero contribution from the term considered. For the second term we close the integration contour in the upper half-plane of the complex variable t with the same result. Therefore we have:

$$a_{an}(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \left[\frac{1}{t - i\pi} - \frac{1}{t + i\pi} \right] = 1. \quad (23)$$

For any finite loop order the expansion structure of the perturbative solution in inverse powers of logarithms ensure the property of the analytic coupling $a_{an}(0) = 1$. The arguments are suitable for all solutions $\Phi(z)$ as long as the singularities are situated at the real axis of the complex z -plane, in particular for the iterative solutions of Refs. [7, 8].

III. EXTRACTION OF THE NONPERTURBATIVE CONTRIBUTIONS

Let us see what the singularities of the integrand of (21) in the complex t -plane are. First of all the integrand has simple poles at $t = \ln x \pm i\pi(1 + 2n)$, $n = 0, 1, 2, \dots$. All the residues of function $\exp(t)/(x + \exp(t))$ at these points are equal to unity. Apart from these poles the integrand of (21) has at $t = \pm i\pi$ poles up to fourth order and logarithmic type branch points which coincide with the poles from the second order to the fourth

order. The initial integration contour in the complex t -plane and singularities of the integrand of Eq. (21) are shown in Fig. 3(a). Let us cut the complex t -plane in a standard way, $t = \pm i\pi - \lambda$, with λ being the real parameter varying from 0 to ∞ . Further on we append the integration by the arch of the "infinite" radius without affecting the value of the integral, and close the integration contour C_1 in the upper half-plane of the complex variable t excluding the singularities at $t = i\pi$. In this case an additional contribution emerges due to the integration along the sides of the cut and around the singularities at $t = i\pi$. The corresponding contour we denote as C_2 . The integration contour C_1 is shown in Fig. 3(b), whereas the contour C_2 is shown in Fig. 3(c).

Let us turn to the integration along the contour C_1 . For the integrand of Eq. (21), which we denote as $F(t)$, the residues at $t = \ln x + i\pi(1 + 2n)$, $n = 0, 1, 2, \dots$ are as follows:

$$\text{Res}F(t) |_{t=\ln x+i\pi(1+2n)} = \Phi(\ln x + 2\pi in) - \Phi(\ln x + 2\pi i(n + 1)). \quad (24)$$

By using the residue theorem one readily obtains the contribution $\Sigma(x)$ to the integral (21) from the integration along the contour C_1

$$\Sigma(x) = \frac{1}{2\pi i} \int_{C_1} F(t) dt = \sum_{n=0}^{\infty} \text{Res}F(t = \ln x + i\pi(1 + 2n)) = \Phi(\ln x) = a(x). \quad (25)$$

One can see that this contribution is exactly equal to the initial Eq. (7). Therefore we call it a perturbative part of $a_{an}(x)$, $a^{pt}(x) = \Sigma(x)$. The remaining contribution of the integral along the contour C_2 can naturally be called a nonperturbative part of $a_{an}(x)$,

$$a_{an}(x) = a^{pt}(x) + a_{an}^{npt}(x). \quad (26)$$

Let us turn to the calculation of $a_{an}^{npt}(x)$. We can omit the terms of the integrand in Eq. (21) which have no singularities at $t = i\pi$. Then we have

$$\begin{aligned} a_{an}^{npt}(x) = & \frac{1}{2\pi i} \int_{C_2} dt \frac{e^t}{x + e^t} \times \left\{ \frac{1}{t - i\pi} - b \frac{\ln(t - i\pi) + C}{(t - i\pi)^2} \right. \\ & + b^2 \left[\frac{[\ln(t - i\pi) + C]^2}{(t - i\pi)^3} - \frac{\ln(t - i\pi) + C}{(t - i\pi)^3} + \frac{\kappa}{(t - i\pi)^3} \right] \\ & \left. - b^3 \left[\frac{[\ln(t - i\pi) + C]^3}{(t - i\pi)^4} - \frac{5}{2} \frac{[\ln(t - i\pi) + C]^2}{(t - i\pi)^4} + (3\kappa + 1) \frac{\ln(t - i\pi) + C}{(t - i\pi)^4} + \frac{\bar{\kappa}}{(t - i\pi)^4} \right] \right\}. \end{aligned} \quad (27)$$

Let us change the variable $t = z + i\pi$ and introduce the function

$$f(z) = \frac{1}{1 - x \exp(-z)}. \quad (28)$$

Then we can rewrite Eq. (27) in the form

$$a_{an}^{npt}(x) = \frac{1}{2\pi i} \int_{\tilde{C}} dz f(z) \left\{ \frac{1}{z} - b \left[\frac{\ln(z)}{z^2} + \frac{C}{z^2} \right] + b^2 \left[\frac{\ln^2(z)}{z^3} + (2C - 1) \frac{\ln z}{z^3} + \frac{\kappa - C + C^2}{z^3} \right] \right\}$$

$$-b^3 \left[\frac{\ln^3(z)}{z^4} + \left(3C - \frac{5}{2}\right) \frac{\ln^2}{z^4} + (3C^2 - 5C + 3\kappa + 1) \frac{\ln z}{z^4} + \frac{C^3 - \frac{5}{2}C^2 + (3\kappa + 1)C + \bar{\kappa}}{z^4} \right] \Bigg\}. \quad (29)$$

The cut in the complex z -plane goes now from zero to $-\infty$. Starting from $z = -\infty - i0$, the contour \tilde{C} goes along the lower side of the cut, then around the origin and then further along the upper side of the cut to $z = -\infty + i0$. x is considered here as a real variable, $x > 1$. Then the contour \tilde{C} can be chosen in such a way that it does not envelop "superfluous" singularities, and the conditions used in the Appendix for finding the corresponding integrals are satisfied. Function (28) with its derivatives

$$\begin{aligned} f'(z) &= -\frac{x \exp(-z)}{(1 - x \exp(-z))^2}, \quad f''(z) = \frac{x \exp(-z)(1 + x \exp(-z))}{(1 - x \exp(-z))^3}, \\ f'''(z) &= -\frac{x \exp(-z)}{(1 - x \exp(-z))^4} \left(1 + 4x \exp(-z) + x^2 \exp(-2z)\right), \\ f''''(z) &= \frac{x \exp(-z)}{(1 - x \exp(-z))^5} \left(1 + 11x \exp(-z) + 11x^2 \exp(-2z) + x^3 \exp(-3z)\right) \end{aligned} \quad (30)$$

decrease exponentially at $z \rightarrow -\infty$. Therefore, we shall omit the boundary terms in formulas given in the Appendix². Then, from Eq. (29) one can obtain

$$\begin{aligned} a_{an}^{npt}(x) = & -\frac{1}{2\pi i} \int_{\tilde{C}} dz \left\{ f'(z) \ln(z) - b \left[(1 + C) \ln(z) + \frac{1}{2} \ln^2(z) \right] f''(z) \right. \\ & + \frac{1}{2} b^2 \left[(2 + \kappa + 2C + C^2) \ln(z) + (1 + C) \ln^2(z) + \frac{1}{3} \ln^3(z) \right] f'''(z) \\ & - \frac{1}{6} b^3 \left[\left(6 + \frac{11}{2} \kappa + \bar{\kappa} + 3(2 + \kappa)C + 3C^2 + C^3\right) \ln z + \frac{3}{2} (2 + \kappa + 2C + C^2) \ln^2 z \right. \\ & \left. \left. + (1 + C) \ln^3 z + \frac{1}{4} \ln^4 z \right] f''''(z) \right\}. \end{aligned} \quad (31)$$

Taking into account that function $f(z)$ with its derivatives is regular at real negative semiaxis of z we can rewrite equation (31) in the form

$$\begin{aligned} a_{an}^{npt}(x) = & -\int_0^{-\infty} du \left\{ f'(u) \bar{\Delta}_1(u) - b \left[(1 + C) \bar{\Delta}_1(u) + \frac{1}{2} \bar{\Delta}_2(u) \right] f''(u) \right. \\ & + \frac{1}{2} b^2 \left[(1 + \kappa + (1 + C)^2) \bar{\Delta}_1(u) + (1 + C) \bar{\Delta}_2(u) + \frac{1}{3} \bar{\Delta}_3(u) \right] f'''(u) \\ & - \frac{1}{6} b^3 \left[\left(2 + \frac{5}{2} \kappa + \bar{\kappa} + 3(1 + C)(1 + \kappa) + (1 + C)^3\right) \bar{\Delta}_1(u) \right. \\ & \left. + \frac{3}{2} (1 + \kappa + (1 + C)^2) \bar{\Delta}_2(u) + (1 + C) \bar{\Delta}_3(u) + \frac{1}{4} \bar{\Delta}_4(u) \right] f''''(u) \Bigg\}, \end{aligned} \quad (32)$$

where u is real, $u < 0$ and $\bar{\Delta}_i(u)$ are discontinuities of the powers of the logarithms

$$\bar{\Delta}_1(u) = \frac{1}{2\pi i} (\ln(u + i0) - \ln(u - i0)) = 1,$$

²Using these formulas with $f = 1$ one can make sure that $a(x = 0) = 1$. The boundary terms should be considered in this case.

$$\begin{aligned}
\bar{\Delta}_2(u) &= \frac{1}{2\pi i} \left(\ln^2(u+i0) - \ln^2(u-i0) \right) = 2\ln(-u), \\
\bar{\Delta}_3(u) &= \frac{1}{2\pi i} \left(\ln^3(u+i0) - \ln^3(u-i0) \right) = 3\ln^2(-u) - \pi^2, \\
\bar{\Delta}_4(u) &= \frac{1}{2\pi i} \left(\ln^4(u+i0) - \ln^4(u-i0) \right) = 4\ln^3(-u) - 4\pi^2\ln(-u).
\end{aligned} \tag{33}$$

Let us introduce the variable $\sigma = \exp(u)$. From Eqs. (30), (32), (33) we obtain

$$\begin{aligned}
a_{an}^{npt}(x) &= -x \int_0^1 d\sigma \left\{ \frac{1}{(x-\sigma)^2} - b \left[1 + C + \ln(-\ln \sigma) \right] \frac{x+\sigma}{(x-\sigma)^3} \right. \\
&\quad + \frac{1}{2}b^2 \left[1 - \frac{\pi^2}{3} + \kappa + (1+C)^2 + 2(1+C)\ln(-\ln \sigma) + \ln^2(-\ln \sigma) \right] \frac{x^2 + 4x\sigma + \sigma^2}{(x-\sigma)^4} \\
&\quad \left. - \frac{1}{6}b^3 \left[2 + \frac{5}{2}\kappa + \bar{\kappa} + 3(1+C)(1 - \frac{\pi^2}{3} + \kappa) + (1+C)^3 + 3 \left(1 - \frac{\pi^2}{3} + \kappa + (1+C)^2 \right) \right. \right. \\
&\quad \left. \left. \times \ln(-\ln \sigma) + 3(1+C)\ln^2(-\ln \sigma) + \ln^3(-\ln \sigma) \right] \frac{x^3 + 11x^2\sigma + 11x\sigma^2 + \sigma^3}{(x-\sigma)^5} \right\}. \tag{34}
\end{aligned}$$

Integrating the terms of Eq. (34) independent of logarithms one can obtain

$$\begin{aligned}
a_{an}^{npt}(x) &= -\frac{1}{x-1} + b \left\{ \frac{(1+C)x}{(x-1)^2} + x \int_0^1 d\sigma \ln(-\ln \sigma) \frac{x+\sigma}{(x-\sigma)^3} \right\} \\
&\quad - \frac{1}{2}b^2 \left\{ \left[1 - \frac{\pi^2}{3} + \kappa + (1+C)^2 \right] \frac{x(x+1)}{(x-1)^3} + x \int_0^1 d\sigma \left[2(1+C)\ln(-\ln \sigma) + \ln^2(-\ln \sigma) \right] \right. \\
&\quad \left. \times \frac{x^2 + 4x\sigma + \sigma^2}{(x-\sigma)^4} \right\} + \frac{1}{6}b^3 \left\{ \left[2 + \frac{5}{2}\kappa + \bar{\kappa} + 3(1+C) \left(1 - \frac{\pi^2}{3} + \kappa \right) + (1+C)^3 \right] \right. \\
&\quad \left. \times \frac{x(x^2 + 4x + 1)}{(x-1)^4} + x \int_0^1 d\sigma \left[3 \left(1 - \frac{\pi^2}{3} + \kappa + (1+C)^2 \right) \ln(-\ln \sigma) \right. \right. \\
&\quad \left. \left. + 3(1+C)\ln^2(-\ln \sigma) + \ln^3(-\ln \sigma) \right] \frac{x^3 + 11x^2\sigma + 11x\sigma^2 + \sigma^3}{(x-\sigma)^5} \right\}. \tag{35}
\end{aligned}$$

This formula gives the nonperturbative contributions in an explicit form.

IV. BEHAVIOR OF THE NONPERTURBATIVE CONTRIBUTIONS AT $Q > \Lambda$

Let us turn to the large Q behavior of the nonperturbative contributions. The following expansions appear to be useful ($x > 1 \geq \sigma \geq 0$)

$$\frac{1}{x-1} = \sum_{n=1}^{\infty} \frac{1}{x^n}, \quad \frac{x}{(x-1)^2} = \sum_{n=1}^{\infty} \frac{n}{x^n}, \quad \frac{x(x+\sigma)}{(x-\sigma)^3} = \sum_{n=1}^{\infty} \frac{n^2\sigma^{n-1}}{x^n},$$

$$\begin{aligned}\frac{x(1+x)}{(x-1)^3} &= \sum_{n=1}^{\infty} \frac{n^2}{x^n}, \quad \frac{x(x^2+4x\sigma+\sigma^2)}{(x-\sigma)^4} = \sum_{n=1}^{\infty} \frac{n^3\sigma^{n-1}}{x^n}, \\ \frac{x(x^2+4x+1)}{(x-1)^4} &= \sum_{n=1}^{\infty} \frac{n^3}{x^n}, \quad \frac{x(x^3+11x^2\sigma+11x\sigma^2+\sigma^3)}{(x-\sigma)^5} = \sum_{n=1}^{\infty} \frac{n^4\sigma^{n-1}}{x^n}.\end{aligned}\quad (36)$$

Note that the coefficients in Eqs. (36) are monomials in powers of n . Expanding Eq. (35) in the inverse powers of x with using Eqs. (36) we have

$$a_{an}^{npt}(x) = \sum_{n=1}^{\infty} \frac{c_n}{x^n}, \quad (37)$$

where

$$\begin{aligned}c_n &= -1 + bn \left\{ 1 + C + n \int_0^1 d\sigma \sigma^{n-1} \ln(-\ln(\sigma)) \right\} \\ &\quad - \frac{1}{2} b^2 n^2 \left\{ 1 + \kappa - \frac{\pi^2}{3} + (1+C)^2 + n \int_0^1 d\sigma \sigma^{n-1} \left[2(1+C) \ln(-\ln(\sigma)) + \ln^2(-\ln(\sigma)) \right] \right\} \\ &\quad + \frac{1}{6} b^3 n^3 \left\{ 2 + \frac{5}{2} \kappa + \bar{\kappa} + 3(1+C) \left(1 - \frac{\pi^2}{3} + \kappa \right) + (1+C)^3 + n \int_0^1 d\sigma \sigma^{n-1} \left[3 \left(1 - \frac{\pi^2}{3} + \kappa \right. \right. \right. \\ &\quad \left. \left. \left. + (1+C)^2 \right) \ln(-\ln \sigma) + 3(1+C) \ln^2(-\ln \sigma) + \ln^3(-\ln \sigma) \right] \right\}.\end{aligned}\quad (38)$$

Making the change of variable $\sigma = \exp(-t)$ and integrating [21], [22] over t one can find

$$\begin{aligned}\int_0^1 d\sigma \sigma^{n-1} \ln(-\ln(\sigma)) &= \int_0^{\infty} dt e^{-nt} \ln(t) = -\frac{1}{n} (\ln(n) + \gamma), \\ \int_0^1 d\sigma \sigma^{n-1} \ln^2(-\ln(\sigma)) &= \int_0^{\infty} dt e^{-nt} \ln^2(t) = \frac{1}{n} \left[(\ln(n) + \gamma)^2 + \frac{\pi^2}{6} \right], \\ \int_0^1 d\sigma \sigma^{n-1} \ln^3(-\ln(\sigma)) &= \int_0^{\infty} dt e^{-nt} \ln^3(t) = -\frac{1}{n} \left[(\ln(n) + \gamma)^3 + \frac{\pi^2}{2} (\ln(n) + \gamma) + 2\zeta_3 \right].\end{aligned}\quad (39)$$

Here γ is the Euler constant, $\gamma \simeq 0.5772$. From Eqs. (38), (39) we finally have

$$\begin{aligned}c_n &= -1 + bn \left[1 + C - \gamma - \ln(n) \right] - \frac{1}{2} b^2 n^2 \left[1 - \frac{\pi^2}{6} + \kappa + \left(1 + C - \gamma - \ln(n) \right)^2 \right] \\ &\quad + \frac{1}{6} b^3 n^3 \left[2 + \frac{5}{2} \kappa + \bar{\kappa} - 2\zeta_3 + \left(1 + C - \gamma - \ln(n) \right)^3 \right. \\ &\quad \left. + 3 \left(1 + C - \gamma - \ln(n) \right) \left(1 - \frac{\pi^2}{6} + \kappa \right) \right].\end{aligned}\quad (40)$$

We can see from Eq. (40) that power series (37) is uniformly convergent at $x > 1$ and its convergence radius is equal to unity. The resulting Eq. (40) is scheme independent in

the sense that n_f dependence is not fixed here, and the method used above allows one in principle to calculate next loops contributions to clarify the general structure of the coefficients c_n .

For numerical evaluation of the coefficients c_n we choose the \overline{MS} scheme values of κ , $\bar{\kappa}$ and assume that $C = 0$. Then the coefficients c_n are dependent on n , n_f , and on the number of loops taken into account. In Table 2 we give the values of c_n and loop corrections for $n_f = 0, 3, 4, 5, 6$. The 1-loop order contributions to c_n are equal to -1 for all n and n_f . Up to 4-loop approximation the coefficients c_n for all n , n_f are negative. With the exception of the 3-loop case at $n_f = 6$, the 2 — 4-loop coefficients c_n for $n_f = 0, 3, 4, 5, 6$ monotonously increase in the absolute value with increasing n . In the ultraviolet region ($x \gg 1$) the nonperturbative contributions are determined by the first term of the series (37). One can see that for all n_f up to four loops c_1 is of the order of unity. The account for the higher loop corrections results in some compensation of the 1-loop leading at large x term of the form $1/x$.

V. MOMENTUM DEPENDENCE OF α_{an}

The expansion coefficients increase in the absolute value not too fast and therefore the representation of the analytic running coupling of QCD in the form

$$\alpha_{an}(Q^2) = \alpha^{pt}(Q^2) + \frac{4\pi}{b_0} \sum_{n=1}^{\infty} c_n \left(\frac{\Lambda^2}{Q^2} \right)^n, \quad (41)$$

with c_n as in Eq. (40) provides one with the effective method for the calculation of α_{an} at $Q > \Lambda$. At that there is no need for the summation of large number of terms of the series. Let us see what the convergence properties of the series (37) are. Since $pn > \ln^3(n)$ for all $n \geq 1$ and $p > p_0 = (3/e)^3 \simeq 1.4$, one can consider the series

$$S_4 = \sum_{n=1}^{\infty} \frac{n^4}{x^n} = \frac{x(x^3 + 11x^2 + 11x + 1)}{(x-1)^5} \quad (42)$$

as a comparison series for Eq. (37) with coefficients (40). The convergence properties of the series Eq. (37) are not worse than that for the series (42). The absolute error for the N -terms approximation of the series (42) is

$$\Delta_4^{(N)} = \frac{1}{x^N(x-1)} \left[\frac{x(x^3 + 11x^2 + 11x + 1)}{(x-1)^4} + \frac{4Nx(x^2 + 4x + 1)}{(x-1)^3} + \frac{6N^2x(x+1)}{(x-1)^2} + \frac{4N^3x}{x-1} + N^4 \right]. \quad (43)$$

It is dependent on Q and N (with given n_f , Λ) and the larger Q and N are, the smaller it is. For rather small $x = 2$ ($Q = 1.4\Lambda$) from Eq. (43) we find that the error of the approximation of the series (42) for $N = 50$ is $\simeq 10^{-9}$ and for $N = 100$ it is $\simeq 10^{-23}$. For larger Q there are no reasons to sum a large number of terms. For the approximation of the

Table 2: The dependence of c_n and loop corrections on n and n_f for the 1 — 4-loop cases.

	n	c_n^{1-loop}	Δ_n^{2-loop}	Δ_n^{3-loop}	Δ_n^{4-loop}	c_n^{2-loop}	c_n^{3-loop}	c_n^{4-loop}
$n_f = 0$	1	-1.0	0.35640	-0.01568	-0.03900	-0.64360	-0.65929	-0.69828
	2	-1.0	-0.45582	0.08741	-0.16455	-1.45582	-1.36841	-1.53296
	3	-1.0	-1.70912	-1.03012	-0.89283	-2.70912	-3.73924	-4.63207
	4	-1.0	-3.24886	-4.51236	-5.11707	-4.24886	-8.76122	-13.87829
	5	-1.0	-5.00160	-11.31238	-18.56032	-6.00160	-17.31398	-35.87430
	6	-1.0	-6.92407	-22.24971	-49.77660	-7.92407	-30.17378	-79.95039
	8	-1.0	-11.17217	-59.34790	-213.31981	-12.17217	-71.52007	-284.83987
	10	-1.0	-15.84626	-120.76945	-616.88776	-16.84626	-137.61571	-754.50348
$n_f = 3$	1	-1.0	0.33405	0.01608	-0.07825	-0.66595	-0.64987	-0.72812
	2	-1.0	-0.42724	0.19624	-0.37379	-1.42724	-1.23101	-1.60480
	3	-1.0	-1.60196	-0.63626	-1.28115	-2.60196	-3.23823	-4.51937
	4	-1.0	-3.04517	-3.48651	-5.07338	-4.04517	-7.53168	-12.60506
	5	-1.0	-4.68801	-9.19185	-16.30462	-5.68801	-14.87987	-31.18449
	6	-1.0	-6.48996	-18.47225	-41.82403	-7.48996	-25.96221	-67.78624
	8	-1.0	-10.47171	-50.22832	-174.16411	-11.47171	-61.70003	-235.86414
	10	-1.0	-14.85275	-103.11451	-499.79465	-15.85275	-118.96725	-618.76190
$n_f = 4$	1	-1.0	0.31252	0.04949	-0.11006	-0.68748	-0.63799	-0.74805
	2	-1.0	-0.39970	0.31341	-0.52880	-1.39970	-1.08630	-1.61510
	3	-1.0	-1.49872	-0.23818	-1.51417	-2.49872	-2.73690	-4.25107
	4	-1.0	-2.84891	-2.48499	-4.77483	-3.84891	-6.33389	-11.10872
	5	-1.0	-4.38587	-7.15989	-13.83276	-5.38587	-12.54576	-26.37852
	6	-1.0	-6.07168	-14.89306	-34.04904	-7.07168	-21.96474	-56.01377
	8	-1.0	-9.79681	-41.69613	-138.28745	-10.79681	-52.49294	-190.78040
	10	-1.0	-13.89549	-86.71011	-394.96378	-14.89549	-101.60560	-496.56937
$n_f = 5$	1	-1.0	0.27813	0.11653	-0.16002	-0.72187	-0.60535	-0.76537
	2	-1.0	-0.35571	0.55755	-0.75021	-1.35571	-0.79817	-1.54837
	3	-1.0	-1.33377	0.50736	-1.78434	-2.33377	-1.82641	-3.61075
	4	-1.0	-2.53536	-0.73077	-4.12859	-3.53536	-4.26613	-8.39472
	5	-1.0	-3.90317	-3.73728	-9.82126	-4.90317	-8.64046	-18.46172
	6	-1.0	-5.40344	-9.01126	-22.11892	-6.40344	-15.41470	-37.53362
	8	-1.0	-8.71859	-28.07388	-85.51994	-9.71859	-37.79247	-123.31240
	10	-1.0	-12.36617	-60.94080	-243.69211	-13.36617	-74.30698	-317.99908
$n_f = 6$	1	-1.0	0.22433	0.25378	-0.22731	-0.77567	-0.52189	-0.74920
	2	-1.0	-0.28692	1.07460	-1.01673	-1.28692	-0.21231	-1.22904
	3	-1.0	-1.07581	1.93179	-2.00536	-2.07581	-0.14402	-2.14938
	4	-1.0	-2.04500	2.37205	-2.96183	-3.04500	-0.67295	-3.63479
	5	-1.0	-3.14826	2.01775	-4.07318	-4.14826	-2.13052	-6.20370
	6	-1.0	-4.35837	0.54419	-6.02095	-5.35837	-4.81418	-10.83512
	8	-1.0	-7.03234	-6.87467	-17.72700	-8.03234	-14.90701	-32.63400
	10	-1.0	-9.97445	-21.85073	-53.77987	-10.97445	-32.82518	-86.60506

analytic running coupling with only one first term of the series (37) for the nonperturbative contributions taken into account,

$$\alpha_{an}(Q^2) \simeq \alpha^{pt}(Q^2) - \frac{4\pi}{b_0} \left\{ 1 - b(1 - \gamma) + \frac{1}{2}b^2 \left(1 - \frac{\pi^2}{6} + \kappa + (1 - \gamma)^2 \right) - \frac{1}{6}b^3 \left[2 + \frac{2}{5}\kappa + \bar{\kappa} - 2\zeta_3 + (1 - \gamma)^3 + 3(1 - \gamma) \left(1 - \frac{\pi^2}{6} + \kappa \right) \right] \right\} \frac{\Lambda^2}{Q^2}, \quad (44)$$

the relative error was studied in Ref. [13] for the 1 — 4-loop order cases. The approximation of the nonperturbative "tail" by the leading term has been shown to give a one percent accuracy for α_{an} already at $Q \sim 5\Lambda$.

In Fig. 4 the x dependencies of a_{an} , a^{pt} , a_{an}^{npt} are presented for the 1 — 4-loop order cases. The nonperturbative contributions have been calculated by the series summation and the analytic running coupling has been calculated through the dispersive representation (10). It turned out that the numerical integration in the cases considered expects definite caution. Insufficient accuracy of integration can look as an ungrounded stability of the analytic running coupling behavior with increase of the order of approximation. The equality (with the accuracy of $2 \cdot 10^{-3}$ percent) of $a_{an}(x)$ calculated through the dispersive representation and the sum of $a^{pt}(x)$ and a_{an}^{npt} calculated as the series for all x from 2 to 20 served us as a criterion for the integration precision. The perturbative component a^{pt} increases with the decrease of x reaching unity at $x \sim 3$ ($Q \sim 1.7\Lambda$). The nonperturbative component is negative (at $x > 1$), it decreases with x compensating for the increase of the perturbative component. According to representation (10) the quantity $a_{an}(x)$ is regular for all $x > 0$ and $a_{an}^{(l)}(0) = 1$ (l is the number of loops of the approximation). Though the derivative of $a_{an}(x)$ is infinite at zero we however make sure numerically of the higher loop stability of $a_{an}(x)$ in the infrared region. As seen in Fig. 4, the 3-loop and 4-loop analytic curves practically coincide even before the normalization at some finite point. As for the corresponding perturbative curves which have no common point at zero, they are not close to each other already at $x < 5$.

Let us consider the momentum dependence of α_{an} (and α^{pt} for comparison) in the low momentum region provided that all solutions are normalized at the central point of the world average value $\alpha(M_Z^2) = 0.1181 \pm 0.002$ [1]. In this case the heavy quark thresholds should be taken into account. It seems natural to demand the analytic running coupling be continuous across thresholds. Let us adopt for α_{an} , α^{pt} and for all 1 — 4-loop order cases the normalization condition $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV and the matching conditions $\alpha^{(n_f=5)}(m_b^2) = \alpha^{(n_f=4)}(m_b^2)$, $m_b = 4.3$ GeV and $\alpha^{(n_f=4)}(m_c^2) = \alpha^{(n_f=3)}(m_c^2)$, $m_c = 1.3$ GeV³. The corresponding sets of parameters Λ are given in Table 3. As seen in Table 3, $\Lambda_{an}^{(n_f=5)} \simeq \Lambda_{pt}^{(n_f=5)}$, since the nonperturbative contributions in fact die out at the scale of normalization⁴. The momentum dependence of α_{an} , α^{pt} for the 1 — 4-loop order cases is presented in Fig. 5. As seen from Fig. 5, the 2 — 4-loop curves for α_{an}

³The matching conditions sensitivity of α_{an} will be considered slightly later.

⁴If to normalize the solutions as $\alpha^{(n_f=4)}(M_\tau^2) = 0.35$, $M_\tau = 1777.03$ MeV [1] with the same matching conditions one obtains substantially larger values of Λ_{an} (e.g., for the 4-loop case $\Lambda_{an}^{(n_f=3,4,5)} \simeq 630$ MeV, 490 MeV, 350 MeV, respectively). At that the higher loop stability is also observed, for the 2 — 4-loop cases $\alpha_{an}^{(n_f=5)}(M_Z^2) = 0.128$.

Table 3: The parameters $\Lambda_{pt}^{(n_f)}$ (MeV), $\Lambda_{an}^{(n_f)}$ (MeV). n_f is the number of active quark flavors, the number of loops is indicated. The normalization and matching conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV, $\alpha^{(n_f=5)}(m_b^2) = \alpha^{(n_f=4)}(m_b^2)$, $m_b = 4.3$ GeV, $\alpha^{(n_f=4)}(m_c^2) = \alpha^{(n_f=3)}(m_c^2)$, $m_c = 1.3$ GeV.

	1-loop	2-loop	3-loop	4-loop
$\Lambda_{pt}^{(n_f=3)}$	143.77	372.50	328.98	332.50
$\Lambda_{pt}^{(n_f=4)}$	120.55	325.91	289.67	291.39
$\Lambda_{pt}^{(n_f=5)}$	88.35	227.51	209.54	209.53
$\Lambda_{an}^{(n_f=3)}$	150.64	454.21	382.30	389.50
$\Lambda_{an}^{(n_f=4)}$	121.61	339.90	298.91	301.64
$\Lambda_{an}^{(n_f=5)}$	88.35	227.60	209.60	209.61

Table 4: The values $\alpha_{an}^{(n_f=4)}(M_\tau^2)$, $\alpha_{pt}^{(n_f=4)}(M_\tau^2)$ with $M_\tau = 1777.03$ MeV. The normalization and matching conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV, $\alpha^{(n_f=5)}(m_b^2) = \alpha^{(n_f=4)}(m_b^2)$, $m_b = 4.3$ GeV.

	1-loop	2-loop	3-loop	4-loop
α_{an}	0.2740	0.2930	0.2943	0.2943
α^{pt}	0.2802	0.3262	0.3179	0.3230

practically coincide indicating the higher loop stability of the analytic running coupling. The corresponding values of $\alpha_{an}^{(n_f=4)}(M_\tau^2)$, $\alpha_{pt}^{(n_f=4)}(M_\tau^2)$ for the 1 — 4-loop cases are given in Table 4.

Note that in Table 4 the values of 3-loop and 4-loop $\alpha_{an}(M_\tau^2)$ are the same and the 4-loop $\alpha_{pt}(M_\tau^2)$ coincides with $\alpha_s(M_\tau)$ of Ref. [2] from τ decays. Thus, the extrapolation to the energy scale M_Z using the 4-loop solution for α_s with 3-loop matching at the bottom quark pole mass $M_b = 4.7$ GeV made in Ref. [2] results in $\alpha_s(M_Z) = 0.1181$. This value is used in our calculations.

Let us consider the threshold matching conditions sensitivity of α_{an} . We fix $\alpha^{(n_f=5)}(M_Z^2)$ at its world average value and vary the value of the matching point μ_b corresponding to the b -quark threshold. Then from the matching condition $\alpha^{(n_f=5)}(\mu_b^2) = \alpha^{(n_f=4)}(\mu_b^2)$ we can find the dependence of the parameters $\Lambda^{(n_f=4)}$ on the matching point value. For a rather wide interval of μ_b it is shown in Fig. 6. We see that $\Lambda^{(n_f=4)}$ parameters for the analytic running coupling go higher than that for the perturbative coupling. In the region of 4 – 5 GeV the μ_b -dependence of $\Lambda_{an}^{(n_f=4)}$ is somewhat weaker than that of $\Lambda_{pt}^{(n_f=4)}$. The dependencies of $\alpha_{an}(M_\tau^2)$, $\alpha^{pt}(M_\tau^2)$ on the matching point μ_b for the 1 — 4-loop order cases are shown in Fig. 7. For the analytic coupling the curves go lower than the corresponding curves for the perturbative coupling. The analytic coupling is much more stable

than the perturbative one with respect to higher loop corrections. In the region of 4 – 5 GeV the μ_b -dependence of $\alpha_{an}(M_\tau^2)$ is considerably weaker than that of $\alpha^{pt}(M_\tau^2)$. In particular, for $\alpha(M_Z^2) = 0.1181$, $m_b = 4.3 \pm 0.2$ GeV

$$\alpha_{an}(M_\tau^2) = 0.2943_{-0.0003}^{+0.0004}, \quad \alpha^{pt}(M_\tau^2) = 0.3230_{-0.0008}^{+0.0008}. \quad (45)$$

We give here the results for the 4-loop α_{an} and α^{pt} .

Let us consider one more heavy quark threshold matching condition. In the framework of the perturbation theory there is the prescription [19] to connect the couplings with different n_f according to which the coupling can be discontinuous at the matching point μ_h . The idea of an implementation of this nontrivial matching conditions is to make the results (e.g., the connection between $\alpha(M_\tau^2)$ and $\alpha(M_Z^2)$) be not substantially dependent on the exact value of the matching point [23]. This conditions take the most simple form for two cases. First, $\mu_h = m_h \equiv m_h(\mu_h)$ where $m_h(\mu)$ is the running \overline{MS} mass of the heavy quark and second, $\mu_h = M_h$ with M_h being the heavy quark pole mass. Choosing the first one according to [19] we have

$$\alpha^{(n_f-1)}(\mu_h^2) = \alpha^{(n_f)}(\mu_h^2) \quad (46)$$

for the 1-loop and 2-loop cases,

$$\alpha^{(n_f-1)}(\mu_h^2) = \alpha^{(n_f)}(\mu_h^2) \left[1 + c_2 \left(\alpha^{(n_f)}(\mu_h^2)/\pi \right)^2 \right] \quad (47)$$

for the 3-loop case, and

$$\alpha^{(n_f-1)}(\mu_h^2) = \alpha^{(n_f)}(\mu_h^2) \left[1 + c_2 \left(\alpha^{(n_f)}(\mu_h^2)/\pi \right)^2 + c_3 \left(\alpha^{(n_f)}(\mu_h^2)/\pi \right)^3 \right] \quad (48)$$

for the 4-loop case. Here

$$c_2 = \frac{11}{72}, \quad c_3 = \frac{564731}{124416} - \frac{82043}{27648}\zeta_3 - \frac{2633}{31104}(n_f - 1). \quad (49)$$

The notations for the coefficients in Eqs. (47) — (49) correspond to Ref. [19]. The coefficients c_n of the present paper have its own definition. For this variant of matching conditions the dependencies of the parameters $\Lambda^{(n_f=4)}$ on the matching point value are close to those shown in Fig. 6 and we do not give the corresponding figure (according to Eq. (46) both matching methods give the same results for the 1, 2-loop order cases). The dependencies of $\alpha_{an}(M_\tau^2)$, $\alpha^{pt}(M_\tau^2)$ on the matching point μ_b for the 3, 4-loop order cases are also close to the previous case of the continuous matching shown in Fig. 7. For $\alpha(M_Z^2) = 0.1181$, $m_b = 4.3 \pm 0.2$ GeV

$$\alpha_{an}(M_\tau^2) = 0.2947_{-0.0003}^{+0.0003}, \quad \alpha^{pt}(M_\tau^2) = 0.3235_{-0.0004}^{+0.0008}. \quad (50)$$

The results are given for the 4-loop case. It is seen from Eqs. (45), (50) that the values of $\alpha_{an}(M_\tau^2)$ for two methods of matching are very close to each other (this is true also for $\alpha^{pt}(M_\tau^2)$).

In Fig. 8 $\alpha_{an}(Q^2)$, $\alpha^{pt}(Q^2)$ are shown for the 1 — 4-loop order cases with the normalization condition $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV and continuous matching at $m_b = 4.3$ GeV, $m_c = 1.3$ GeV. Without going into the details, we give in this figure the data from Table 6 of Ref. [2] for the world summary of measurements of α_s .

VI. CONCLUSIONS

In contrast to the recent papers [24] we apply the analytic approach to the perturbative QCD running coupling constant in the form of the standard expansion in the inverse powers of logarithms up to the four loop order. An introduction of the complex variable t in the spectral representation for the analytic running coupling and study of the singularities structure of the integrand allowed one to divide the analytic running coupling into perturbative (initial) component and nonperturbative one (appeared as a consequence of "forced" analyticity) exactly, as it is illustrated in Fig. 3. These components turned out to be connected with different singularities in the complex t -plane. It is shown that the nonperturbative contributions can be represented in the form of the expansion Eq. (37) in inverse powers of the momentum squared where the coefficients c_n are defined by Eq. (40). Eq. (41) gives the effective method non-connected with numerical integration for calculation of the analytic running coupling at $Q > \Lambda$ with the calculation accuracy of standard mathematical functions. It can be important for making popular the considered variant of α_{an} . In practice, for Q corresponding to $n_f = 5$, it is sufficient to take account of the leading nonperturbative term, as in Eq. (44).

On the basis of the developed method we study the momentum dependence of α_{an} giving at the same time the behavior of the perturbative running coupling. To fix the solutions we used for all of them the same normalization condition at M_Z where the nonperturbative contributions are negligible quantities. We can see in Fig. 4, Fig. 5, Fig. 7, and Fig. 8 the higher loop stability of the analytic running coupling for all $Q > 0$ (the 1-loop case falls out of the common picture). For the perturbative case the higher loop stability takes place only at sufficiently large Q .

We considered two variants of heavy quark threshold matching conditions for α_{an} . The results appeared to be very similar. We showed the b -quark threshold matching conditions stability of the analytic running coupling⁵. As a criterion we considered the dependence on the matching point μ_b of the correspondence of $\alpha_{an}(M_\tau^2)$ to $\alpha_{an}(M_Z^2)$ for the 1 — 4-loop cases (for comparison α^{pt} was considered simultaneously). The situation is illustrated by Fig. 7. The energy scale evolution of the analytic running coupling gives $\alpha(M_\tau^2) = 0.2943_{-0.0003}^{+0.0004}$ for the normalization at the world average value of $\alpha(M_Z^2) = 0.1181$ and matching by continuity with $m_b = 4.3 \pm 0.2$ GeV. With the same normalization condition at the scale of M_Z for both matching methods considered $\alpha_{an}(M_\tau^2)$ is about 0.03 less than $\alpha^{pt}(M_\tau^2)$. Therefore, if one regards α_{an} as a true running coupling constant the noticeable discrepancy with τ lepton decay data [1, 2] arises.

A possible solution of this problem can be found if one changes the normalization condition at M_Z . Let it corresponds to the value of Refs. [25, 26] $\alpha(M_Z^2) \simeq 0.124$ which is appreciably larger than the conventional one. Then for the 4-loop case the result is $\alpha_{an}(M_\tau^2) = 0.3270_{-0.0003}^{+0.0004}$ with continuous matching at $m_b = 4.3 \pm 0.2$ GeV.

As seen in Fig. 8, the analytic approach gives the running coupling which does not deviate essentially at sufficiently large momentum values from the usual perturbative running coupling constant. In the infrared region this approach allows one to solve the principal difficulty connected with nonphysical singularities. The question arises whether the approach described takes into account the nonperturbative contributions to the right degree.

⁵The c -quark threshold matching was considered in Ref. [12] with same result.

There is a whole series of the approaches in which the nonperturbative contributions to the Green functions and running coupling are studied. These approaches are beyond the scope of the present paper and we only point out some papers [27, 28, 29, 30, 31, 32] dealing with the nonperturbative contributions to the running coupling.

To summarize, the analytic running coupling seems to be a good basis for the problem of "genuine nonperturbative" contributions in "physical" α_s , which needs further analysis.

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APPENDIX

We give here the identities we need in our computations which can be obtained by means of an integration by parts. Let function $f(z)$ of complex variable z be regular in some domain D where $z = 0 \in D$. Dealing with the singularities of the integrands at the origin of the pole type coinciding with the logarithmic type branch points we cut the domain D along real negative semiaxis. Then for any contour \tilde{C} in the cut domain \tilde{D} which goes from $z_1 \neq 0$ to $z_2 \neq 0$ one can find

$$\int_{\tilde{C}} \frac{dz}{z} f(z) = - \int_{\tilde{C}} dz \ln(z) f'(z) + \ln(z) f(z) \Big|_{z_1}^{z_2},$$

$$\int_{\tilde{C}} \frac{dz}{z^2} f(z) = - \int_{\tilde{C}} dz \ln(z) f''(z) + \left\{ -\frac{1}{z} f(z) + \ln(z) f'(z) \right\} \Big|_{z_1}^{z_2},$$

$$\int_{\tilde{C}} \frac{dz}{z^3} f(z) = -\frac{1}{2} \int_{\tilde{C}} dz \ln(z) f'''(z) + \left\{ -\frac{1}{2z^2} f(z) - \frac{1}{2z} f'(z) + \frac{1}{2} \ln(z) f''(z) \right\} \Big|_{z_1}^{z_2},$$

$$\int_{\tilde{C}} \frac{dz}{z^4} f(z) = -\frac{1}{6} \int_{\tilde{C}} dz \ln(z) f''''(z) + \left\{ -\frac{1}{3z^3} f(z) - \frac{1}{6z^2} f'(z) - \frac{1}{6z} f''(z) + \frac{1}{6} \ln(z) f'''(z) \right\} \Big|_{z_1}^{z_2},$$

$$\int_{\tilde{C}} \frac{dz}{z} \ln(z) f(z) = -\frac{1}{2} \int_{\tilde{C}} dz \ln^2(z) f'(z) + \frac{1}{2} \ln^2(z) f(z) \Big|_{z_1}^{z_2},$$

$$\begin{aligned} \int_{\tilde{C}} \frac{dz}{z^2} \ln(z) f(z) = & - \int_{\tilde{C}} dz \left(\ln(z) + \frac{1}{2} \ln^2(z) \right) f''(z) + \left\{ -\frac{1}{z} f(z) - \frac{\ln(z)}{z} f(z) + \ln(z) f'(z) \right. \\ & \left. + \frac{1}{2} \ln^2(z) f'(z) \right\} \Big|_{z_1}^{z_2}, \end{aligned}$$

$$\int_{\tilde{C}} \frac{dz}{z^3} \ln(z) f(z) = - \int_{\tilde{C}} dz \left(\frac{3}{4} \ln(z) + \frac{1}{4} \ln^2(z) \right) f'''(z) + \left\{ -\frac{1}{4z^2} f(z) - \frac{3}{4z} f'(z) - \frac{\ln(z)}{2z^2} f(z) \right.$$

$$\begin{aligned}
& -\frac{\ln(z)}{2z}f'(z) + \frac{3\ln(z)}{4}f''(z) + \frac{\ln^2(z)}{4}f'''(z) \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^4} \ln(z)f(z) = - \int_{\tilde{C}} dz \left(\frac{11}{36} \ln(z) + \frac{1}{12} \ln^2(z) \right) f''''(z) + \left\{ -\frac{1}{9z^3}f(z) - \frac{5}{36z^2}f'(z) \right. \\
& \left. - \frac{11}{36z}f''(z) - \frac{\ln(z)}{3z^3}f(z) - \frac{\ln(z)}{6z^2}f'(z) - \frac{\ln(z)}{6z}f''(z) + \frac{11\ln(z)}{36}f'''(z) + \frac{\ln^2(z)}{12}f''''(z) \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z} \ln^2(z)f(z) = -\frac{1}{3} \int_{\tilde{C}} dz \ln^3(z)f'(z) + \frac{1}{3} \ln^3(z)f(z) \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^2} \ln^2(z)f(z) = - \int_{\tilde{C}} dz \left(2\ln(z) + \ln^2(z) + \frac{1}{3}\ln^3(z) \right) f''(z) + \left\{ -\frac{2}{z}f(z) - \frac{2\ln(z)}{z}f'(z) \right. \\
& \left. + 2\ln(z)f'(z) - \frac{\ln^2(z)}{z}f(z) + \ln^2(z)f'(z) + \frac{\ln^3(z)}{3}f''(z) \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^3} \ln^2(z)f(z) = - \int_{\tilde{C}} dz \left(\frac{7}{4}\ln(z) + \frac{3}{4}\ln^2(z) + \frac{1}{6}\ln^3(z) \right) f'''(z) + \left\{ -\frac{1}{4z^2}f(z) \right. \\
& \left. - \frac{7}{4z}f'(z) - \frac{\ln(z)}{2z^2}f(z) - \frac{3\ln(z)}{2z}f'(z) + \frac{7\ln(z)}{4}f''(z) \right. \\
& \left. - \frac{\ln^2(z)}{2z^2}f(z) - \frac{\ln^2(z)}{2z}f'(z) + \frac{3\ln^2(z)}{4}f''(z) + \frac{\ln^3(z)}{6}f'''(z) \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^4} \ln^2(z)f(z) = - \int_{\tilde{C}} dz \left(\frac{85}{108}\ln(z) + \frac{11}{36}\ln^2(z) + \frac{1}{18}\ln^3(z) \right) f''''(z) + \left\{ -\frac{2}{27z^3}f(z) \right. \\
& - \frac{19}{108z^2}f'(z) - \frac{85}{108z}f''(z) - \frac{2\ln(z)}{9z^3}f(z) - \frac{5\ln(z)}{18z^2}f'(z) - \frac{11\ln(z)}{18z}f''(z) + \frac{85\ln(z)}{108}f'''(z) \\
& \left. - \frac{\ln^2(z)}{3z^3}f(z) - \frac{\ln^2(z)}{6z^2}f'(z) - \frac{\ln^2(z)}{6z}f''(z) + \frac{11\ln^2(z)}{36}f'''(z) + \frac{\ln^3(z)}{18}f''''(z) \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z} \ln^3(z)f(z) = -\frac{1}{4} \int_{\tilde{C}} dz \ln^4(z)f'(z) + \frac{1}{4} \ln^4(z)f(z) \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^2} \ln^3(z)f(z) = - \int_{\tilde{C}} dz \left(6\ln(z) + 3\ln^2(z) + \ln^3(z) + \frac{1}{4}\ln^4(z) \right) f''(z) \\
& + \left\{ -\frac{6}{z}f(z) - \frac{6\ln(z)}{z}f'(z) + 6\ln(z)f'(z) - \frac{3\ln^2(z)}{z}f(z) \right. \\
& \left. + 3\ln^2(z)f'(z) - \frac{\ln^3(z)}{z}f(z) + \ln^3(z)f'(z) + \frac{\ln^4(z)}{4}f''(z) \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^3} \ln^3(z)f(z) = - \int_{\tilde{C}} dz \left(\frac{45}{8}\ln(z) + \frac{21}{8}\ln^2(z) + \frac{3}{4}\ln^3(z) + \frac{1}{8}\ln^4(z) \right) f'''(z)
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -\frac{3}{8z^2}f(z) - \frac{45}{8z}f'(z) - \frac{3\ln(z)}{4z^2}f(z) - \frac{21\ln(z)}{4z}f'(z) + \frac{45\ln(z)}{8}f''(z) - \frac{3\ln^2(z)}{4z^2}f(z) \right. \\
& - \frac{9\ln^2(z)}{4z}f'(z) + \frac{21\ln^2(z)}{8}f''(z) - \frac{\ln^3(z)}{2z^2}f(z) - \frac{\ln^3(z)}{2z}f'(z) + \frac{3\ln^3(z)}{4}f''(z) + \frac{\ln^4(z)}{8}f'''(z) \left. \right\} \Big|_{z_1}^{z_2}, \\
& \int_{\tilde{C}} \frac{dz}{z^4} \ln^3(z) f(z) = - \int_{\tilde{C}} dz \left(\frac{575}{216} \ln(z) + \frac{85}{72} \ln^2(z) + \frac{11}{36} \ln^3(z) + \frac{1}{24} \ln^4(z) \right) f''''(z) \\
& + \left\{ -\frac{2}{27z^3}f(z) - \frac{65}{216z^2}f'(z) - \frac{575}{216z}f''(z) - \frac{2\ln(z)}{9z^3}f(z) - \frac{19\ln(z)}{36z^2}f'(z) - \frac{85\ln(z)}{36z}f''(z) \right. \\
& + \frac{575\ln(z)}{216}f'''(z) - \frac{\ln^2(z)}{3z^3}f(z) - \frac{5\ln^2(z)}{12z^2}f'(z) - \frac{11\ln^2(z)}{12z}f''(z) + \frac{85\ln^2(z)}{72}f'''(z) - \frac{\ln^3(z)}{3z^3}f(z) \\
& \left. - \frac{\ln^3(z)}{6z^2}f'(z) - \frac{\ln^3(z)}{6z}f''(z) + \frac{11\ln^3(z)}{36}f'''(z) + \frac{\ln^4(z)}{24}f''''(z) \right\} \Big|_{z_1}^{z_2}.
\end{aligned}$$

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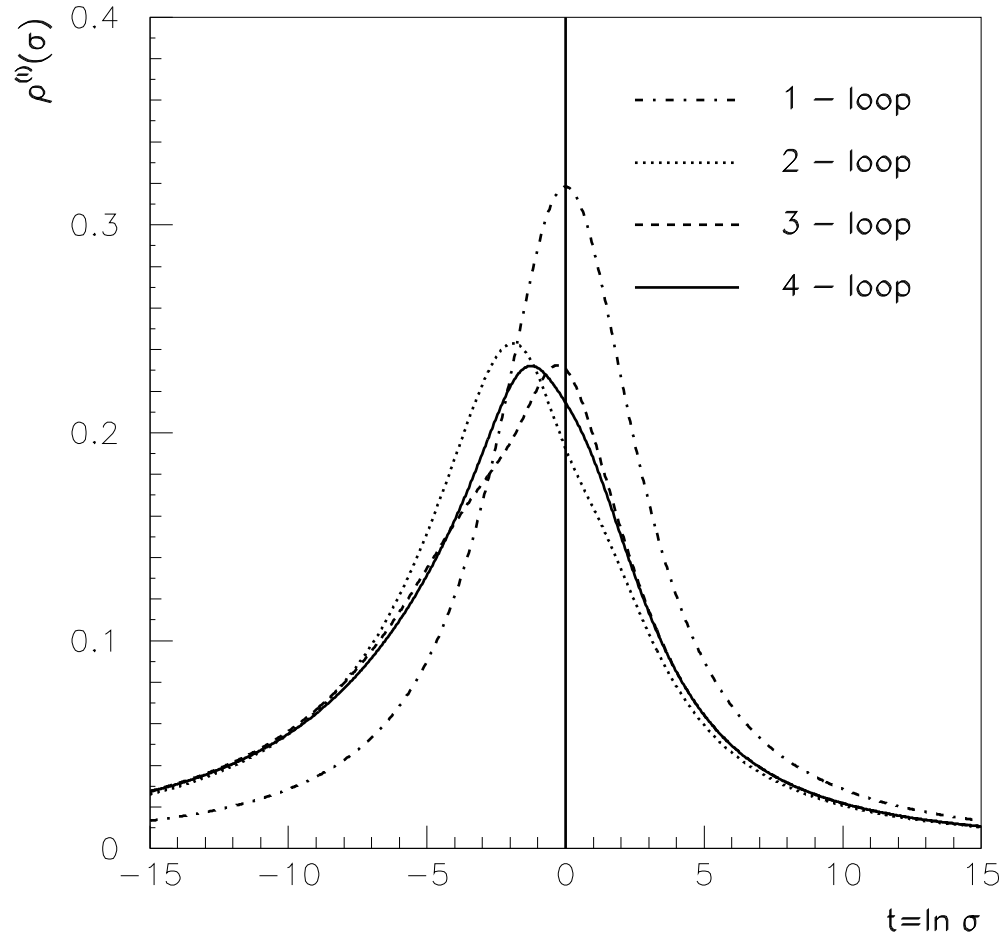


Figure 1: The spectral density of the analytic running coupling up to four loop order.

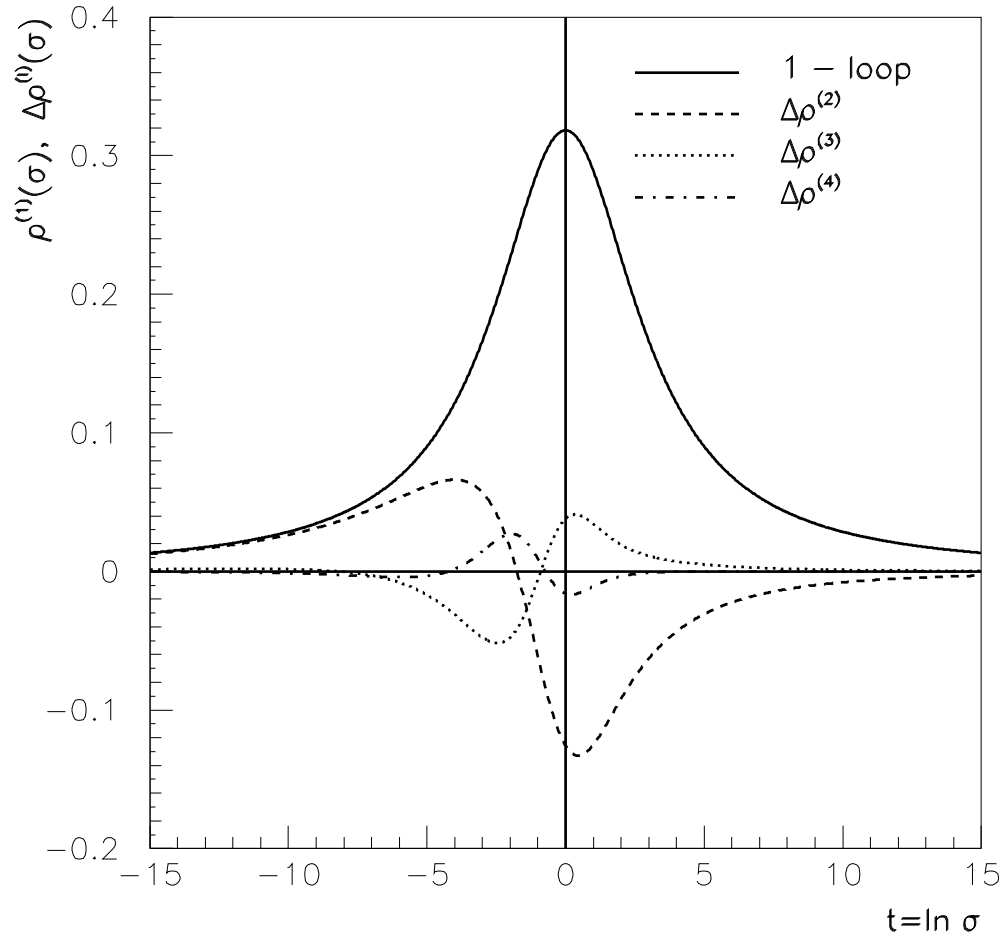


Figure 2: The higher loop order corrections for the spectral density.

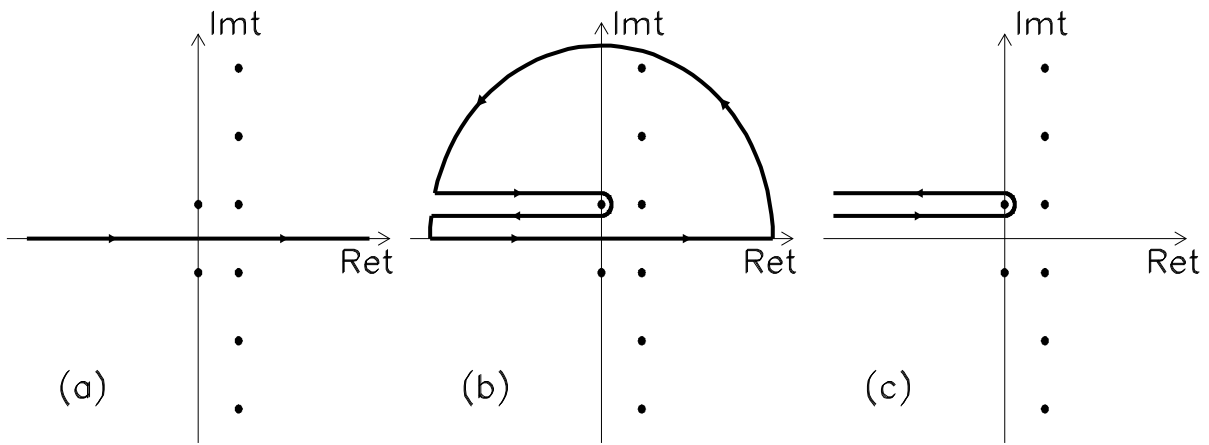


Figure 3: Complex t -plane integration. Perturbative contributions arise from the poles at $t = \ln x + i\pi(1 + 2n)$, $n = 0, 1, 2, \dots$. Nonperturbative contributions emerge from the singularities at $t = i\pi$.

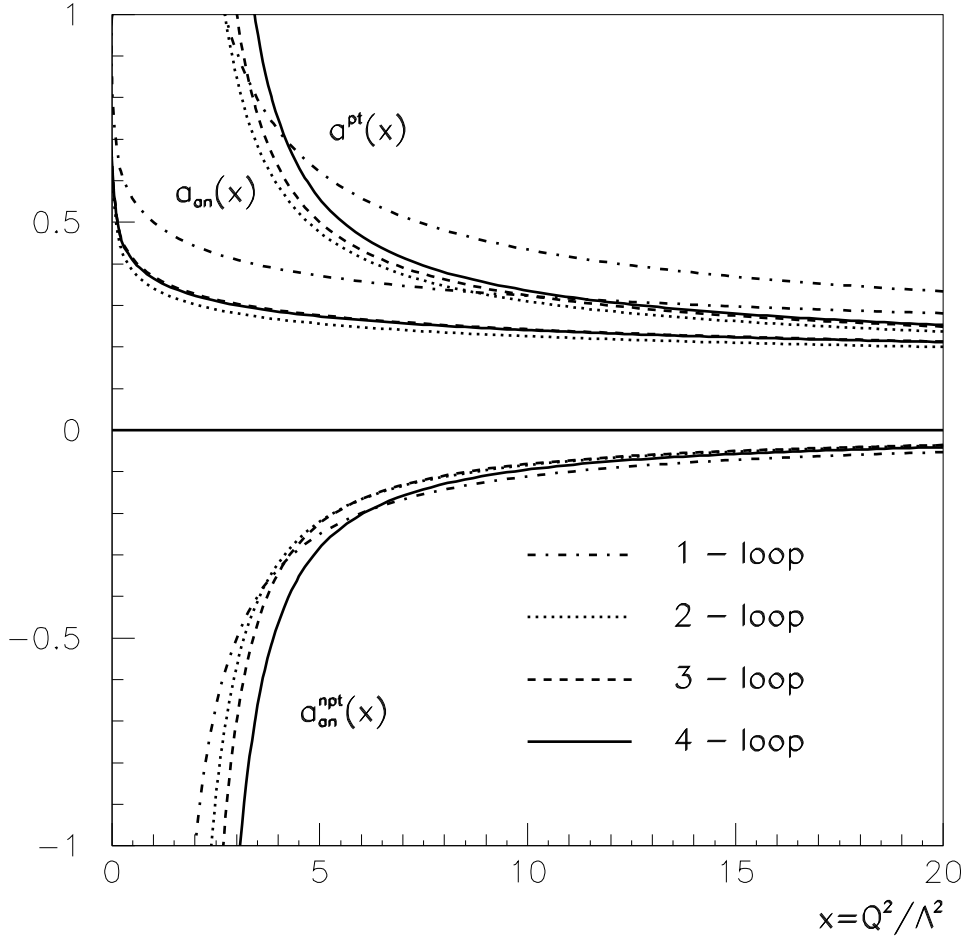


Figure 4: The analytic running coupling a_{an} and its perturbative component a^{pt} and nonperturbative component a_{an}^{np} as functions of $x = Q^2/\Lambda^2$ for the 1 — 4-loop order cases. Here $n_f = 3$.

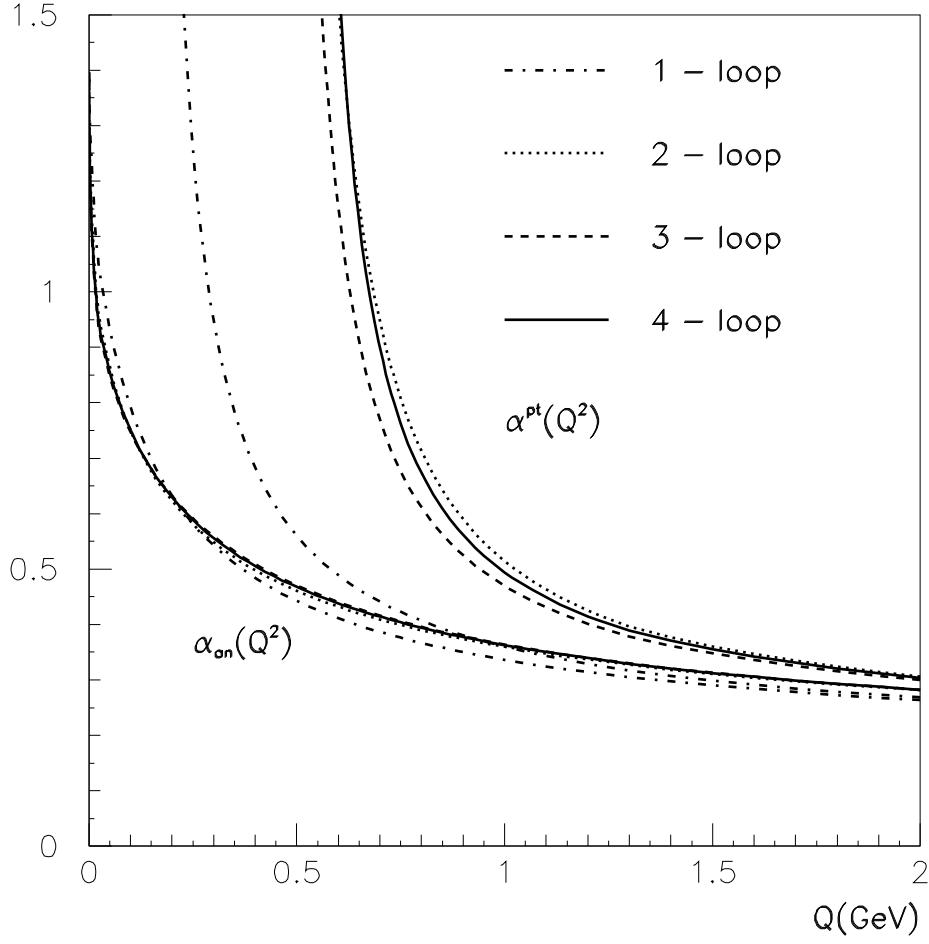


Figure 5: The momentum dependence of α_{an} , α^{pt} for the 1 — 4-loop order cases. The normalization and matching conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV; $\alpha^{(n_f=5)}(m_b^2) = \alpha^{(n_f=4)}(m_b^2)$, $m_b = 4.3$ GeV; $\alpha^{(n_f=4)}(m_c^2) = \alpha^{(n_f=3)}(m_c^2)$, $m_c = 1.3$ GeV.

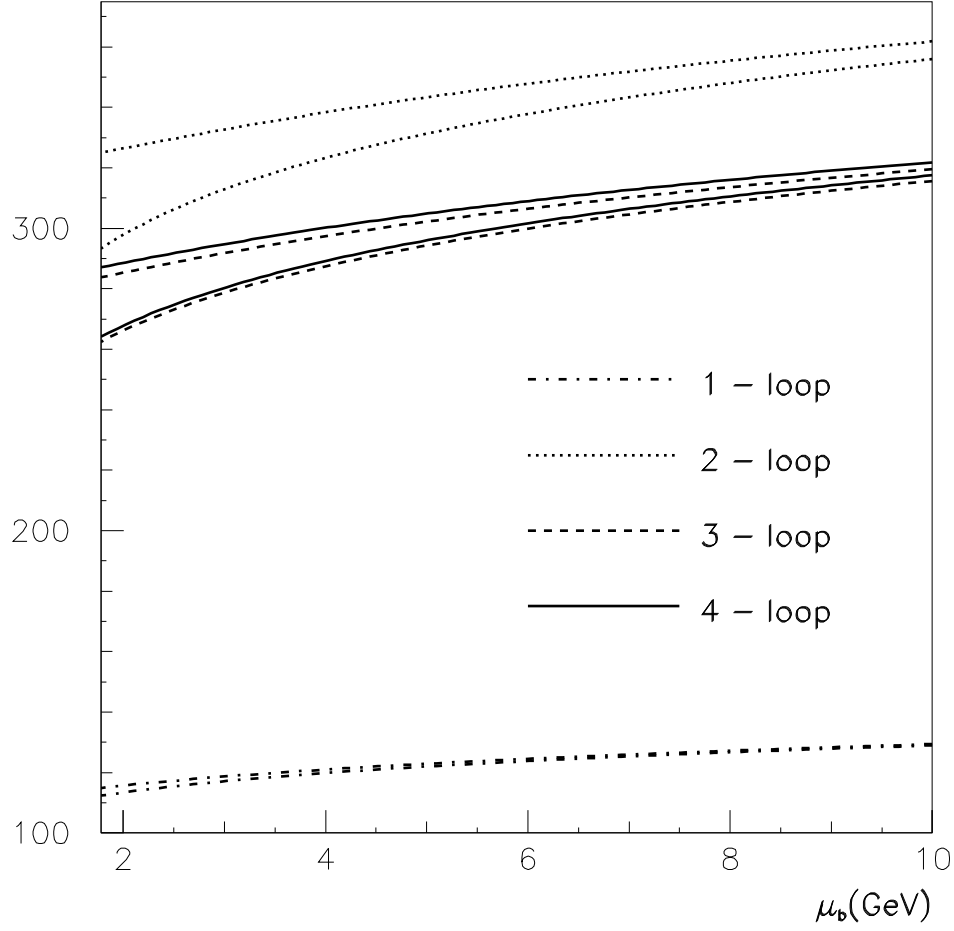


Figure 6: The dependencies of $\Lambda_{an}^{(n_f=4)}$, $\Lambda_{pt}^{(n_f=4)}$ on the matching point μ_b for the 1 — 4-loop order cases. For the analytic coupling the curves go above the corresponding curves for the perturbative coupling. The normalization and matching conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV; $\alpha^{(n_f=5)}(\mu_b^2) = \alpha^{(n_f=4)}(\mu_b^2)$.

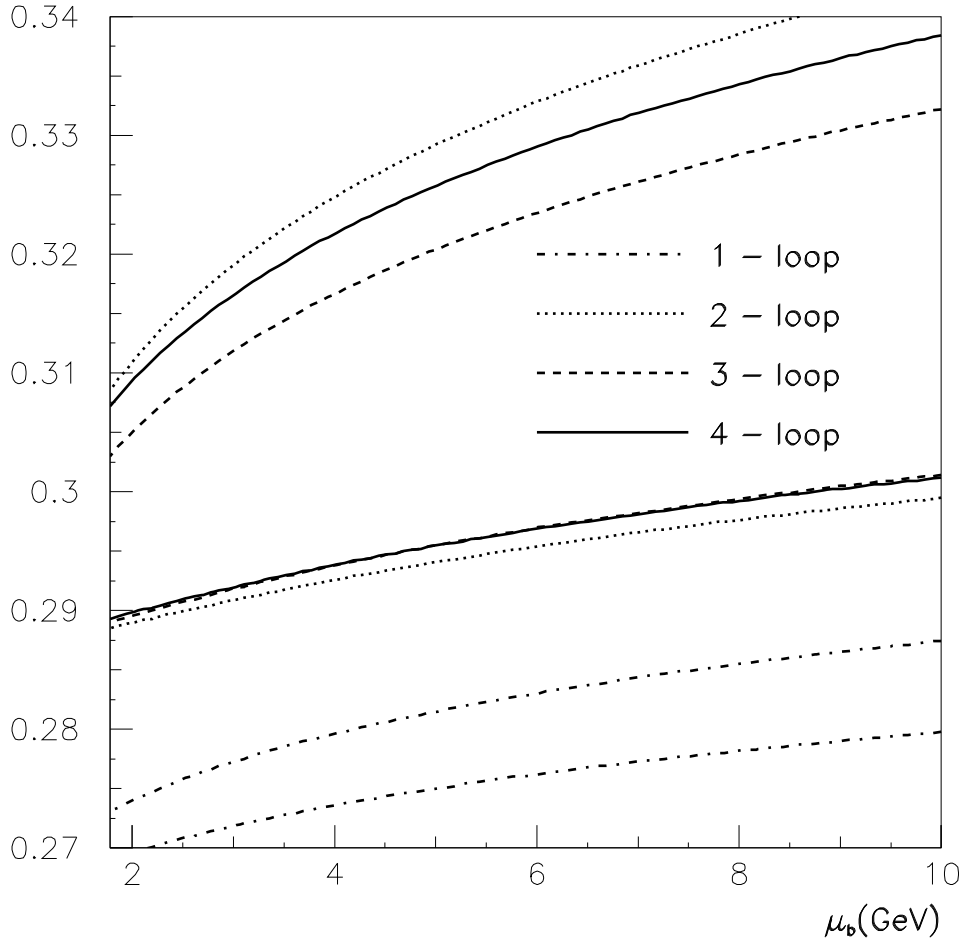


Figure 7: The dependencies of $\alpha_{an}(M_\tau^2)$, $\alpha^{pt}(M_\tau^2)$ on the matching point μ_b for the 1 — 4-loop order cases. For the analytic coupling the curves go lower than the corresponding curves for the perturbative coupling. The normalization and matching conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV; $\alpha^{(n_f=5)}(\mu_b^2) = \alpha^{(n_f=4)}(\mu_b^2)$.

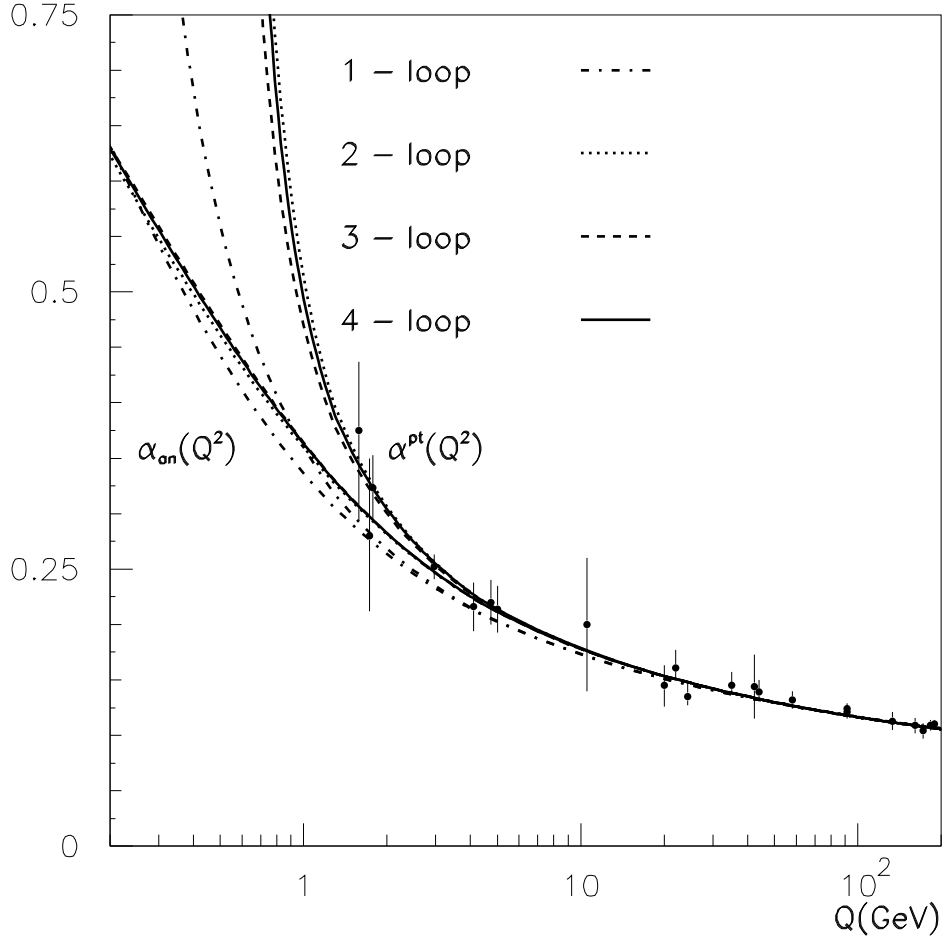


Figure 8: The analytic and perturbative couplings $\alpha_{an}(Q^2)$, $\alpha^{pt}(Q^2)$ for the 1 — 4-loop order cases. The normalization conditions are $\alpha^{(n_f=5)}(M_Z^2) = 0.1181$, $M_Z = 91.1882$ GeV; $\alpha^{(n_f=5)}(m_b^2) = \alpha^{(n_f=4)}(m_b^2)$, $m_b = 4.3$ GeV; $\alpha^{(n_f=4)}(m_c^2) = \alpha^{(n_f=3)}(m_c^2)$, $m_c = 1.3$ GeV.