The two fundamental differential equations of electrostatics are,

$$\vec{\nabla} \bullet \mathbf{E} = \rho / \varepsilon_0 \qquad [I.1]$$

$$\vec{\nabla} \times \mathbf{E} = \mathbf{0} \qquad [I.2]$$

Note that [I.2] implies,

$$\exists \Phi : \mathbf{E} = -\vec{\nabla}\Phi \qquad [I.3]$$

Putting [I.3] into [I.1],

$$\rho / \varepsilon_0 = \vec{\nabla} \bullet (-\vec{\nabla}\Phi) = -\vec{\nabla}^2\Phi \qquad [I.4]$$

We know [I.4] as Laplace's equation. It has the solution,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \cdot d^3\mathbf{x}' \qquad [I.5]$$

Now, introduce the so-called "a-potential"; it is *defined* as "offset" from the potential $\Phi(\mathbf{x})$ by "a", and we construct this potential just like the integral definition [I.5] as,

$$\Phi_a(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{\sqrt{(\mathbf{x} - \mathbf{x}')^2 + a^2}} \cdot d^3\mathbf{x}' \qquad [I.6]$$

We can clearly see,

$$\lim_{a \to 0} \left(\Phi_a(\mathbf{x}) \right) = \Phi(\mathbf{x}) \qquad [I.7]$$

Take the Laplacian of both sides of [I.6], and "expect" it to be equal to something "analogous to" $-\frac{\rho}{\varepsilon_0}$,

$$\nabla^2\Phi_a(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \nabla^2 \left(\frac{\rho(\mathbf{x}')}{\sqrt{(\mathbf{x} - \mathbf{x}')^2 + a^2}} \right) \cdot d^3\mathbf{x}' \overset{???}{=} "-\frac{\rho'}{\varepsilon_0}" \qquad [I.8]$$

The Laplacian only affects non-integration variables ("all the unprimed stuff"), since the $\mathbf{x}'$ is a counting device for the integral and not an argument of Φ_a as $\Phi_a(\mathbf{x})$.

Thus, we need the radial part of the Laplacian only,

$$\vec{\nabla}_r^{\,2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \cdot) \qquad [I.9]$$

Using [I.9] on $\frac{1}{\sqrt{(\mathbf{x}-\mathbf{x}')^2 + a^2}} = \frac{1}{\sqrt{r^2 + a^2}}$, we get,

$$\nabla_r^2\left(\frac{1}{\sqrt{(\mathbf{x}-\mathbf{x}')^2+a^2}}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left(\frac{1}{\sqrt{r^2+a^2}}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right)\left(\frac{-\frac{1}{2}\cdot(2r+0)}{(r^2+a^2)^{3/2}}\right)=\frac{\frac{\partial}{\partial r}\left(\frac{-r^3}{(r^2+a^2)^{3/2}}\right)}{r^2}$$
$$=\frac{-\left(\frac{3r^2}{(r^2+a^2)^{3/2}}+\frac{r^3(-\frac{3}{2})(2r+0)}{(r^2+a^2)^{5/2}}\right)}{r^2}=\frac{-\left(\frac{3}{(r^2+a^2)^{3/2}}+\frac{-3r^2}{(r^2+a^2)^{5/2}}\right)}{r^2}=\frac{3}{(r^2+a^2)^{5/2}}-\frac{3}{r^2(r^2+a^2)^{3/2}} \quad \text{[I.10]}$$

Combining the two remaining terms $\frac{3}{(r^2+a^2)^{5/2}}$ and $\frac{3}{r^2(r^2+a^2)^{3/2}}$,

$$\nabla_r^2\left(\frac{1}{\sqrt{(\mathbf{x}-\mathbf{x}')^2+a^2}}\right)=\frac{3r^2}{r^2(r^2+a^2)^{5/2}}-\frac{3(r^2+a^2)^{1/2}}{r^2(r^2+a^2)^{5/2}}=3\frac{r^2-(r^2+a^2)^{1/2}}{r^2(r^2+a^2)^{5/2}} \quad \text{[I.11]}$$

Jackson claims,

$$\nabla_r^2\left(\frac{1}{\sqrt{(\mathbf{x}-\mathbf{x}')^2+a^2}}\right)\overset{????}{=}\frac{3a^2}{(r^2+a^2)^{5/2}} \quad \text{[I.12]}$$

Perhaps we made a math error somewhere. Let's try Maple, and entrust *it* to carry out $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\frac{1}{\sqrt{r^2+a^2}}$, as in [I.10]. The derivative is,

```
> 'diff(((r^2)*diff((r^2 + a^2)^(-1/2),r)),r)/(r^2)'=diff(((r^2)*diff((r^2
+ a^2)^(-1/2),r)),r)/(r^2);
```

$$\frac{\frac{\partial}{\partial r}\left(r^2\left(\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+a^2}}\right)\right)\right)}{r^2}=\frac{-\frac{3\,r^2}{(r^2+a^2)^{(3/2)}}+\frac{3\,r^4}{(r^2+a^2)^{(5/2)}}}{r^2} \quad \text{[I.13]}$$

Maybe I just carried out my parenthesis wrong somewhere. Better get good at that before class begins! Now, this *does* simplify to the result Jackson has!

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> (-3*r^2/(r^2+a^2)^(3/2)+3*r^4/(r^2+a^2)^(5/2))/r^2=simplify((-
3*r^2/(r^2+a^2)^(3/2)+3*r^4/(r^2+a^2)^(5/2))/r^2);
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$$\frac{-\frac{3\,r^2}{(r^2+a^2)^{(3/2)}}+\frac{3\,r^4}{(r^2+a^2)^{(5/2)}}}{r^2}=-\frac{3\,a^2}{(r^2+a^2)^{(5/2)}} \quad \text{[I.14]}$$

This [I.14] confirms [I.12], and so [I.8] becomes,

$$\nabla^2\Phi_a(\mathbf{x})=-\frac{1}{4\pi\varepsilon_0}\int\rho(\mathbf{x}')\left(\frac{3a^2}{(r^2+a^2)^{5/2}}\right)\cdot d^3\mathbf{x}' \quad \text{[I.15]}$$

$$\int_0^R\int_0^{\pi/2}\int_0^{\pi}\rho(\mathbf{x}')\cdot d^3\mathbf{x}'=\int_0^R\left(\rho(r,\theta',\phi')+\tfrac{1}{6}r^2\vec{\nabla}^2\rho+\ldots\right)\cdot 4\pi r^2\cdot dr \quad \text{[I.16]}$$

…and in [I.15] we have the abbreviation $r = |\mathbf{x} - \mathbf{x}'|$. Now, consider what happens if a → 0 and r → 0, the function $\frac{3a^2}{\sqrt{a^2+r^2}}$ becomes infinite, but it is otherwise-well behaved. The limits a → 0 and r → 0 are not artificial, as they correspond to a point charge.

Also, consider the integral of the function $\frac{3a^2}{(a^2+r^2)^{5/2}}$,

$$\int_0^R \int_0^{\pi/2} \int_0^{\pi} \frac{3a^2}{(a^2+r^2)^{5/2}} \cdot dr \cdot r \cdot d\theta \cdot r \sin\theta \cdot d\phi = 12\pi a^2 \int_0^R \frac{r^2}{(a^2+r^2)^{5/2}} \cdot dr \qquad \text{[I.17]}$$

```
> 'int(((r^2)/(a^2+r^2)^(5/2)),r)'=int(((r^2)/(a^2+r^2)^(5/2)),r);
```

$$\int \frac{r^2}{(r^2+a^2)^{(5/2)}}\, dr = \frac{r^3}{3\,(r^2+a^2)^{(3/2)}\, a^2}$$

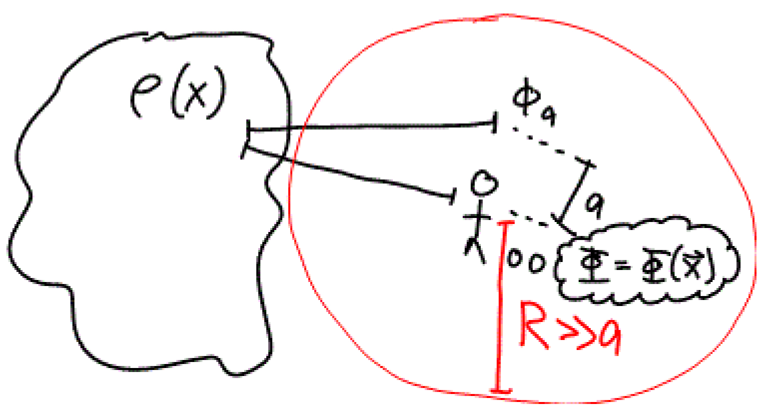
[I.18]

Rearranging [I.18] and evaluating it at the limits

$$\left(\frac{r^3}{3a^2(r^2+a^2)^{3/2}} \right)_{r=0}^{r=R} = \left(\frac{R^3}{3a^2(R^2+a^2)^{3/2}} - \frac{r^3}{3a^2(r^2+a^2)^{3/2}} \right) \qquad \text{[I.19]}$$

???? Jackson claims the bracketed-expressionhas a volume integral of 4π for arbitrary "a"! I don't think so…

Now, imagine a sphere about the point whre we are considering the $\Phi(\mathbf{x})$ as in [I.5],



[I.20]

Some things to notice about [I.20],

1) $R >> a$

2) R is chosen such that $\rho(\mathbf{x}')$ *changes little* over the interior of the sphere (to be completely general, there *is* some $\rho(\mathbf{x})$ inside the sphere of [I.20] even so we didn't draw it so).

For a well behaved potential [I.5] (physically *expected*), the contribution to the integral-Laplacian defining the "a-potential", [I.8], will vanish like a^2 as a → 0 on the <u>outside</u> of the sphere. Thus, <u>we only need to integrate the inside of the sphere</u>.

Now, we can Taylor-expand about $\mathbf{x}' = \mathbf{x}$ the well-behaved $\rho(\mathbf{x}')$, as,

$$\rho(\mathbf{x}') = \rho(\mathbf{x}) + \tfrac{1}{6} r^2 \nabla^2 \rho + ... \qquad \text{[I.21]}$$

NOW…if we can justify [I.21] and [I.19], we can justify the Dirac-delta solution to Poisson's equation.

Okay, concerning [I.19], let the lower bound of integration be replaced by a small number, ε

$$\begin{aligned}\left(\frac{r^3}{3a^2(r^2+a^2)^{3/2}}\right)_{r=\varepsilon}^{r=R} &= \left(\frac{R^3}{3a^2(R^2+a^2)^{3/2}} - \frac{\varepsilon^3}{3a^2(\varepsilon^2+a^2)^{3/2}}\right) = \frac{R^3}{3a^2R^3(1+\frac{a^2}{R^2})^{3/2}} - \frac{\varepsilon^3}{3a^2a^3(\frac{\varepsilon^2}{a^2}+1)^{3/2}} \\ &= \frac{1}{3a^2(1+\frac{a^2}{R^2})^{3/2}} - \frac{\varepsilon^3}{3a^5(1)^{3/2}} = \frac{R^3}{3a^5(\frac{a^2}{R^2}+1)^{3/2}} - 0 = \frac{R^3}{3a^5(\sqrt{\frac{a^2}{R^2}+1})^3}\end{aligned} \qquad \text{[I.22]}$$

Recall the binomial series expansion,

$$\sqrt{1+x} \equiv \sum_{n=0}^{n\to\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + ... \qquad \text{[I.23]}$$

In [I.20], we agreed by design that $R >> a$, and so the denominator of [I.22], $\sqrt{\frac{a^2}{R^2}+1}$, becomes,

$$\left(\frac{r^3}{3a^2(r^2+a^2)^{3/2}}\right)_{r=\varepsilon}^{r=R} = \frac{R^3}{3a^5\left((1+\frac{1}{2}(\frac{a^2}{R^2})-...)^3 + O(\frac{a^{12}}{R^{12}})\right)} = \frac{R^3}{3a^5(1+\frac{1}{4}\frac{a^4}{R^4}+\frac{a^2}{R^2})(1+\frac{1}{2}(\frac{a^2}{R^2}))} = \frac{R^3}{3a^5(1+\frac{a^2}{R^2})(1+\frac{1}{2}\frac{a^2}{R^2})} \qquad \text{[I.24]}$$

…in which we have dropped the terms $\frac{1}{4}\frac{a^4}{R^4}$ and higher. Then, use $(1+x)(1+\frac{1}{2}x) = 1+\frac{3}{2}x+\frac{1}{2}x^2$ in [I.24] to get,

$$\left(\frac{r^3}{3a^2(r^2+a^2)^{3/2}}\right)_{r=\varepsilon}^{r=R} = \frac{R^3}{3a^5(1+\frac{3}{2}\frac{a^2}{R^2}+\frac{1}{2}\frac{a^4}{R^4})} = \frac{R^3}{3a^5(1+\frac{3}{2}\frac{a^2}{R^2})} \qquad \text{[I.25]}$$

Put [I.25] into [I.17],

$$12\pi a^2 \int_0^R \frac{r^2}{(a^2+r^2)^{5/2}} \cdot dr = 12\pi a^2 \frac{R^3}{3a^5(1+\frac{3}{2}\frac{a^2}{R^2})} = 4\pi \frac{R^3}{a^3(1+\frac{3}{2}\frac{a^2}{R^2})} = 4\pi \frac{x^{-3}}{(1+\frac{3}{2}x^2)} \qquad \text{[I.26]}$$

Aggg…failure.