GLIDE REFLECTION MATRIX

Let R be the rotation about the origin by $\pi/4$, and G be the glide-reflection that first reflects a point across the line $y = x$ and then translates the result by $(2, 2)$. Describe $R \circ G$ and $G \circ R$. If either is a reflection find the mirror line. If either is a glide-reflection, find both the mirror line and the translation.

Recall that the general matrix for a rotation around the origin by an angle θ (counter-clockwise) is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

Therefore,

$$R(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta=\frac{\pi}{4}} \mathbf{x} \tag{2}$$

$$R(\mathbf{x}) = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \mathbf{x} \tag{3}$$

Reflection across the line $y = x$ can be expressed in the matrix

$$G(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{4}$$

It appears then that

$$\begin{aligned}
(R \circ G)(\mathbf{x}) &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) && (5) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_2 + 2 \\ x_1 + 2 \end{bmatrix} && (6) \\
&= \begin{bmatrix} -\sqrt{2}\left(\frac{1}{2}x_1 - \frac{1}{2}x_2\right) \\ \sqrt{2}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + 2\right) \end{bmatrix} && (7) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_1 \\ \frac{1}{2}\sqrt{2}x_1 + \frac{1}{2}\sqrt{2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (8) \\
&= \begin{bmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (9)
\end{aligned}$$

The $2x2$ matrix does not look like a rotation (the $(1, 2)$ and $(2, 1)$ entries don't add up to zero). It looks like a reflection matrix, around the origin, where the angle of incidence of the mirror line is $3\pi/8$. However, the translation is not parallel to the mirror line. This is not a glide-reflection. Still, there should be no fixed points. I'll check..

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (10) \\
&= \begin{bmatrix} -\sqrt{2}\left(\frac{1}{2}x_1 - \frac{1}{2}x_2\right) \\ \sqrt{2}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + 2\right) \end{bmatrix} && (11) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_1 \\ \frac{1}{2}\sqrt{2}x_1 + \frac{1}{2}\sqrt{2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (12) \\
\mathbf{0} &= \begin{bmatrix} \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_1 \\ \frac{1}{2}\sqrt{2}x_1 + \frac{1}{2}\sqrt{2}x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (13) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2}x_2 - \frac{1}{2}\sqrt{2}x_1 - x_1 \\ \frac{1}{2}\sqrt{2}x_1 - x_2 + \frac{1}{2}\sqrt{2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (14) \\
&= \begin{bmatrix} -1 - \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -1 + \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} && (15)
\end{aligned}$$

Now I can construct an augmented matrix.

$$\begin{bmatrix} -1 - \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & -1 + \frac{1}{2}\sqrt{2} & 2\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 - \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

which is an inconsistent system, as expected.

See the file "0702000 Glide Reflection Matrix-A.gsp". It supports my conclusions about the mirror line, but when it comes to the translation at the end, the measurements aren't what I expected. I don't know how to account for this.

Now try the other composition

$$\begin{aligned}
(G \circ R)(\mathbf{x}) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (17) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (18)
\end{aligned}$$

This looks again like a glide reflection, with an angle of incidence of $\pi/8$. Once again the glide is not parallel to the mirror line, so it isn't a glide reflection.

Now I will look for fixed points (there should not be any).

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (19) \\
\mathbf{0} &= \begin{bmatrix} \sqrt{2}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \\ \sqrt{2}\left(\frac{1}{2}x_1 - \frac{1}{2}x_2\right) \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (20) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2}x_1 - x_1 + \frac{1}{2}\sqrt{2}x_2 \\ \frac{1}{2}\sqrt{2}x_1 - x_2 - \frac{1}{2}\sqrt{2}x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (21) \\
&= \begin{bmatrix} \frac{1}{2}\sqrt{2} - 1 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -1 - \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} && (22)
\end{aligned}$$

Now for the augmented matrix.

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} - 1 & \frac{1}{2}\sqrt{2} & 2 \\ \frac{1}{2}\sqrt{2} & -1 - \frac{1}{2}\sqrt{2} & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\sqrt{2} - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

which again is an inconsistent system, as expected. See the GSP file "0702000 Glide Reflection Matrix-B.gsp". The measurements do support my calculations this time.