of these integrals, the pole at $s=1$. (Poles at $s=3$ and beyond will have subdominant behavior.) Thus, we need only consider

$$\operatorname*{Res}_{s=1}\left[R^{-s/2}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{2\sqrt{\pi}(1-s)}\right]=\frac{\log R}{2\sqrt{R}}+\frac{\log 2}{\sqrt{R}}$$

$$\operatorname*{Res}_{s=1}\left[-R^{-s/2}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{2\sqrt{\pi}(1-s)^2}\right]=-\frac{\log^2 R}{8\sqrt{R}}-\frac{\log 2\log R}{2\sqrt{R}}-\frac{\pi^2+6\log^2 2}{12\sqrt{R}}$$

The above residues were evaluated by noting that the pole at $s=1$ is a double pole in the former integrand and a triple pole in the latter integrand. Now we may take these results and plug them into the limit expression for the original integral:

$$\begin{aligned}\int_0^1 dx\frac{\log^2 x}{\sqrt{x(1-x)}} &= \lim_{R\to\infty}\left(-2\pi\int_0^R dx\frac{\log x}{\sqrt{x(1+x)}}+\pi\log^2 R-\frac{\pi^3}{3}\right)\\
&= \lim_{R\to\infty}\left(-2\pi\left[2\sqrt{R}\log R\int_0^1\frac{dt}{\sqrt{1+Rt^2}}+4\sqrt{R}\int_0^1 dt\frac{\log t}{\sqrt{1+Rt^2}}\right]+\pi\log^2 R-\frac{\pi^3}{3}\right)\\
&= \lim_{R\to\infty}\left(-2\pi\left[2\sqrt{R}\log R\left(\frac{\log R}{2\sqrt{R}}+\frac{\log 2}{\sqrt{R}}\right)+4\sqrt{R}\left(-\frac{\log^2 R}{8\sqrt{R}}-\frac{\log 2\log R}{2\sqrt{R}}-\frac{\pi^2+6\log^2 2}{12\sqrt{R}}\right)\right]+\pi\log^2 R-\frac{\pi^3}{3}\right)\\
&= \lim_{R\to\infty}\left(-2\pi\log^2 R-4\pi\log 2\log R+\pi\log^2 R+4\pi\log 2\log R+\frac{2\pi^3+12\pi\log^2 2}{3}+\pi\log^2 R-\frac{\pi^3}{3}\right)\end{aligned}$$

Note that all terms depending on R cancel, and we finally have

$$\int_0^1 dx\frac{\log^2 x}{\sqrt{x(1-x)}}=\frac{\pi^3}{3}+4\pi\log^2 2$$

I know this was a long read, but I hope by introducing the Mellin transform approach to determining the limit we needed, the approach ended up expressing the original integral in terms of other integrals that could be evaluated – or in this case, approximated, more simply. I think it led to completely new places to which I have never been, and I hope this leads the way to more interesting integral evaluations using complex analysis.

4 COMMENTS

jsgordonblog wrote:
December 11, 2017 at 1:21 am

I like dogs.

Ron wrote:
December 11, 2017 at 1:50 am

I like taking dog-arithms.

Frank wrote:
December 23, 2017 at 9:53 pm

There's an error with your latex, I don't think it recognizes \operatorname.

Ron wrote:
December 27, 2017 at 5:39 am

Frank, thanks for writing in. It is the Mathjax server more than anything – it will render everything after a few reloads. I have been trying to understand the problem but at this point all I can say is, keep reloading until the inline equations and the operatorname stuff renders properly. It does, eventually.