(a) As before, we can write the potential as a sum of terms $R(\rho)Q(\phi)Z(z)$. In this problem there is no ϕ dependence, so $Q = 1$. Also, the boundary conditions on Z are that it vanish at ∞ and be finite at 0, whence $Z(z) \propto \exp(-kz)$ for any k. Then the potential expansion becomes

$$\Phi(\rho, z) = \int_0^\infty A(k) e^{-kz} J_0(k\rho)\, dk. \tag{6}$$

To evaluate the coefficients $A(k)$, we multiply both sides by $\rho J_0(k'\rho)$ and integrate over ρ at $z = 0$:

$$\begin{aligned}\int_0^\infty \rho\Phi(\rho, 0) J_0(k'\rho)\, d\rho &= \int_0^\infty A(k) \left\{ \int_0^\infty \rho J_0(k\rho) J_0(k'\rho)\, d\rho \right\} dk \\ &= \frac{A(k')}{k'}\end{aligned}$$

so

$$\begin{aligned}A(k) &= k \int_0^\infty \rho\Phi(\rho, 0) J_0(k\rho)\, d\rho \\ &= kV \int_0^a \rho J_0(k\rho) d\rho.\end{aligned}$$

Plugging this back into (6),

$$\Phi(\rho, z) = V \int_0^\infty \int_0^a k\rho' e^{-kz} J_0(k\rho) J_0(k\rho')\, d\rho'\, dk. \tag{7}$$

The ρ' integral can be done right away. To do it, I appealed to the differential equation for J_0:

$$J_0''(u) + \frac{1}{u} J_0'(u) + J_0(u) = 0$$

so

$$\begin{aligned}\int_0^x u J_0(u)\, du &= -\int_0^x u J_0''\, du - \int_0^x J_0'(u)\, du \\ &= -\left| u J_0'(u) \right|_0^x + \int_0^x J_0'(u)\, du - \int_0^x J_0'(u)\, du \\ &= -\left| u J_0'(u) \right|_0^x = -x J_0'(x) = x J_1(x).\end{aligned}$$

(In going from the first to second line, I integrated by parts.) Then (7) becomes

$$\Phi(\rho, z) = aV \int_0^\infty J_1(ka) J_0(k\rho) e^{-kz}\, dk. \tag{8}$$