Together with this physical picture one can give an intuitively appealing, if simplistic, interpretation of the probability distributions for spontaneous and stimulated emission. The crux of this interpretation is the concept of a fundamental probability for the black hole to jump down one level in the chain associated with emission in a given mode. We call it $p$. It must depend on the mode, but we shall assume it is identical for spontaneous and stimulated scattering. The probability for spontaneous emission of $n$ quanta in the given mode must equal the probability of $n$ successive jumps, $p^n$, times the probability no further jump takes place, $1 - p$. This is of the same form as our earlier result (4.11) with the identification $p = e^{-\beta}$.

Now consider stimulated emission due to $n$ incident quanta. The probability that the black hole makes exactly $j_i$ jumps down due to the influence of the $i$th incident quantum is $(1 - e^{-\beta})e^{-\beta j_i}$. The incident quanta are assumed to act independently, so we multiply these factors to get $(1 - e^{-\beta})^n e^{-\beta j}$, where $j$ is the sum of all the $j_i$. We must also multiply this factor by the number of ways that $j$ indistinguishable emitted quanta can be partitioned into $n$ classes, one for each of $n$ indistinguishable incident quanta, in order to take care of Bose–Einstein statistics. We get exactly the expression (4.24) for the probability of stimulated emission of $j$ quanta due to $n$ incident ones.

Thus our simple viewpoint provides a unified treatment of spontaneous and stimulated emission in terms of a single transition probability $e^{-\beta}$. From this point of view the fundamental quantity is $e^{-\beta}$, not $\Gamma$. (Scattering is described by its own scattering probability $1 - \Gamma_0$.) Only a deeper theory will be able to ascertain the value of the approach presented here, but it is certainly a suggestive one.

## Black Hole Mass Distribution

We are accustomed to regard a black hole in a stationary state as having definite $M$, $Q$, and $L$. But the Hawking radiation denies this comfortable idea. Since the black hole emits quanta according to a probability distribution, the change in the hole's mass (and charge and angular momentum) is not known sharply, and neither can the $M$, $Q$, and $L$ after emission. A radiating black hole must thus be described by a statistical distribution over $M$, $Q$, and $L$ with certain dispersions. Yet all theoretical work indicates that it makes sense to think of $M$, $Q$, and $L$ as well defined. Hence the distribution must be a sharp one about mean values $\bar{M}$, $\bar{Q}$, $\bar{L}$ that will play the role of effective mass, charge, and angular momentum of the hole.

It is clearly of interest to find out what the distribution looks like. This is possible by way of a slight reformulation of Wheeler's dictum "a black hole has no hair"[50] to conform with the fact that $M$, $Q$, and $L$ are not known precisely. There is little doubt that the principle must be held to stipulate that $\bar{M}$, $\bar{Q}$, and $\bar{L}$ are the only parameters of the probability distribution that determine external properties of the hole. Let us then deduce the form of the distribution. To keep things simple I shall only consider the case with $\bar{Q} = \bar{L} = 0$ and concentrate on the (marginal) distribution for mass alone.

We designate the distribution by $P(M,\bar{M})$ and regard it as continuous for convenience. It must satisfy

$$\int_0^\infty P(M,\bar{M})dM = 1 , \qquad (4.40)$$

$$\int_0^\infty MP(M,\overline{M})dM = \overline{M} \ . \tag{4.41}$$

Let the black hole radiate over a short time and emit a representative sample of quanta in all modes (including different species of particles) according to the distributions $p_{sp}(n)$ like (4.11) and its fermion analog.[51] The new distribution of the hole's mass after emission shall be given by the composite distribution

$$\widetilde{P} = \sum_{\{n\}} P(M + \sum_i n_i\varepsilon_i, \overline{M}) \prod_i p_{sp}(n_i) \ , \tag{4.42}$$

where the product is over all modes $i$ emitted, and the sum over $\{n\}$ means a sum over all possible sets of occupation numbers $n_i$. In (4.42) we take the probability that the hole originally has an excess energy $\sum_i n_i\varepsilon_i$ over $M$, multiply it by the probability (the $\prod p_{sp}$) of one possible event by which it may radiate just this excess, and sum over all possible radiation events, and over all possible excess energies. Now let us expand $P$ in (4.42) in powers of $\sum_i n_i\varepsilon_i$ to second order. We get

$$\widetilde{P} = P(M,\overline{M}) + \frac{\partial}{\partial M} P(M,\overline{M}) \cdot \sum_i \overline{n}_i\varepsilon_i + \tfrac{1}{2} \frac{\partial^2}{\partial M^2} P(M,\overline{M}) \cdot \sum_{ij} \overline{n_i n_j}\varepsilon_i\varepsilon_j \ , \tag{4.43}$$

where bars indicate averaging with respect to the distribution $p_{sp}$. Since $\overline{n_i n_j} = \overline{n}_i\overline{n}_j$ for $i \neq j$, (4.43) can be put in the more convenient form [also correct to $O(n_i^2\varepsilon_i^2)$]

$$\widetilde{P} = P(M - \Delta\overline{M}, \overline{M}) + \tfrac{1}{2} \frac{\partial^2}{\partial M^2} P(M,\overline{M}) \cdot s^2 \ , \tag{4.44}$$

where

$$\Delta\overline{M} = -\sum_i \overline{n}_i\varepsilon_i \ , \tag{4.45}$$

$$s^2 = \sum_i (\overline{n_i^2} - \overline{n}_i^2)\varepsilon_i^2 \ . \tag{4.46}$$

Clearly $\Delta\overline{M}$ is the change in the mean mass of the hole, while $s^2$ is the variance of the emitted energy, which must equal the change in the variance

$$\sigma^2 \equiv \int_0^\infty (M - \overline{M})^2 P(M,\overline{M})dM \ , \tag{4.47}$$

of $M$. By assumption $\sigma^2$ can only depend on $\overline{M}$, so we can write $s^2 = (d\sigma^2/d\overline{M})\Delta\overline{M}$ to first order. Also by assumption $\widetilde{P}$ can only be $P(M, \overline{M} + \Delta\overline{M})$ since $\overline{M}$ determines the distribution entirely. Making the identifications in (4.44) and expanding to first order in $\Delta\overline{M}$, we get the equation