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A LAGRANGIAN FORMULATION OF THE CLASSICAL AND QUANTUM DYNAMICS OF SPINNING PARTICLES

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A spinning particle is described in terms of its position $\phi^\mu(\tau)$ and of an additional spin degree of freedom $\psi^\mu(\tau)$ which is an odd element of a Grassmann algebra. Its motion is described by an action which is invariant under both general reparametrizations and local supergauge transformations. For a particular realization of the canonical commutation relations we obtain a first quantized version of the Dirac equation in an analogous fashion to the way that the Klein-Gordon equation arises from the line element Lagrangian for a spinless particle. This procedure is extended to include internal symmetries and in this case the physical states turn out to be singlets under the group.

1. Introduction

The motion of a relativistic string is described by the non-linear Nambu-Goto Lagrangian which is proportional to the area of the world sheet swept out by the string in space-time [1]. This Lagrangian is the natural extension to the string of the line element Lagrangian which describes the motion of a spinless pointlike particle. More realistic dual models [2] have been constructed where the world sheet of a string is described not only by its position $\phi^\mu(\tau, \sigma)$ but also by a fermionic spinor field $\psi^\mu(\tau, \sigma)$ which is an odd element of a Grassmann algebra. Zumino [3] has used the formalism of superspace to construct a non-linear Lagrangian for the

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Neveu-Schwarz-Ramond model that gives both the equations of motion and the constraints.

The possibility of using Grassmann algebras in quantum mechanics has already been pointed out by several authors [4, 5] as a way of introducing spin degrees of freedom at the classical level, and in this paper we employ this method to represent a spinning point particle. As in the case of the spinning string we introduce a fermionic variable $\psi^\mu(\tau)$ to partner the position variable $\phi^\mu(\tau)$. The variable $\psi^\mu(\tau)$ is an odd element of a Grassmann algebra and takes care of the spin degrees of freedom. τ is any parameter along the world line of the particle and μ is the Lorentz index. The motion of the spinning particle is described by an action which is invariant under both arbitrary reparametrizations and local supersymmetry transformations. The invariance under τ reparametrizations is required by the fact that we must be able to choose any parameter without altering the physics of the system [6], whilst the supergauge invariance is necessary to ensure that negative norm states, which may be induced by the time component of ψ^μ, do not appear in the physical spectrum. The quantum theory of this system can easily be constructed by imposing the canonical commutation relations and we find that, in addition to furnishing a description of a Dirac particle, there is an alternative realization which leads to a bosonic spectrum, consisting of one vector and two scalar states. As we have pointed out in a separate paper together with Deser and Zumino [7] our Lagrangian also describes the interaction of 'supergravity' [8] and supermatter in one time and no space dimensions and may therefore be a good starting point for such a system in four dimensions.

The organization of the paper is as follows: in sect. 2 we describe in some detail the case of a spinless particle in a way which is appropriate to generalize to the spinning case; in sect. 3 we construct an action for the massless spinning case invariant under reparametrizations and local supersymmetry; in sect. 4 we quantize the system and exhibit both the fermionic and bosonic realizations of the commutation relations; in sect. 5 we include in our Lagrangian minimal couplings to external electromagnetic and gravitational fields; in sect. 6 we treat the massive case following the ideas of Berezin and Marinov [9] whose work reached us after the completion of the massless case *; finally, in sect. 7, we treat the case of an internal O(2) symmetry starting from a complete reparametrization invariant action in a superspace with co-ordinates $Z^M = (\tau, \theta^1, \theta^2)$ where $\theta^m (m = 1,2)$ are odd elements of a Grassmann algebra, and we find that the resultant bosonic states are O(2) singlets, indicating that we have introduced a colour symmetry.

* These authors do not, however, discuss the full supersymmetry of the system.

2. Spinless point-like particle

The motion of a free spinless point-like particle of mass m is described by the following action;

$$S = m \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{\phi}^2} , \tag{2.1}$$

which represents the integral along the world line of the particle between the points $\phi^\mu(\tau_i)$ and $\phi^\mu(\tau_f)$ where τ is an arbitrary parameter. The action is invariant under transformations of the form

$$\tau \to \tau' = f(\tau) , \tag{2.2}$$

with f an arbitrary function of τ. The equation of motion of the particle is given by

$$\frac{d}{d\tau} p^\mu(\tau) = 0 , \tag{2.3}$$

where

$$p_\mu = \frac{\partial L}{\partial \dot{\phi}^\mu}$$

is the momentum canonically conjugate to ϕ. As a consequence of the invariance of the action we have the following primary constraint;

$$p^2 - m^2 = 0 , \tag{2.4}$$

which is, of course, just the mass-shell condition. It is useful to work in the proper-time gauge specified by

$$p^\mu = m\dot{\phi}^\mu , \tag{2.5}$$

whereupon the equation of motion becomes

$$\ddot{\phi}^\mu = 0 , \tag{2.6}$$

with solution

$$\phi^\mu = q^\mu + p^\mu \tau . \tag{2.7}$$

Finally the constraint becomes

$$\dot{\phi}^2 = 1 . \tag{2.8}$$

The equation of motion (2.6) may be derived from the linearized Lagrangian

$$L = \tfrac{1}{2} m^2 \dot{\phi}^2 , \tag{2.9}$$

which is now only invariant under translations of τ. In order to extend the above procedure to the spinning case, we rewrite the previous Lagrangian in a different way. The Lagrangian (2.9) is just the Klein-Gordon Lagrangian in one time and no space dimensions if we now consider ϕ to be a field. It is well-known that the Klein-Gordon Lagrangian may be made generally covariant by introducing the metric tensor $g_{\mu\nu}$ or, equivalently, the vierbein field e^a_μ where a is a flat index and μ a curved index and

$$e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu} , \tag{2.10}$$

where η_{ab} is the flat metric. In an arbitrary number of dimensions we have

$$L = -\tfrac{1}{2} e\, e^\mu_a e^\nu_b \partial_\mu \phi \partial_\nu \phi \eta^{ab} , \tag{2.11}$$

where $e \equiv \det e^a_\mu$. It is easy to show that the corresponding action is invariant under any general co-ordinate transformation $x \to x' = x - f(x)$ if

$$\delta e^a_\mu = f^\lambda \partial_\lambda e^a_\mu + \partial_\mu f^\lambda e^a_\lambda ,$$

$$\delta \phi = f^\lambda \partial_\lambda \phi . \tag{2.12}$$

In one dimension $e^a_\mu \equiv e$ and we have

$$\delta e = f\dot{e} + \dot{f}e ,$$

$$\delta \phi = f\dot{\phi} . \tag{2.13}$$

We will refer to e as the vierbein field although einbein would be more appropriate.

A 'generally covariant' Lagrangian in one dimension is now given by

$$L = \frac{1}{2}\left(\frac{\dot{\phi}^2}{e} + m^2 e\right) . \tag{2.14}$$

The 'cosmological' term $m^2 e$ clearly does not spoil reparametrization invariance and has been introduced to give a finite mass to the particle. The action corresponding to (2.14) is different from (2.1) in that we now have two fields e and ϕ which are to be varied independently. It is straightforward to verify that this action is equivalent to the previous formulation by using the Euler-Lagrange equation for e and substituting back into the Lagrangian (2.14). From (2.14) we have the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{\phi}}{e}\right) = 0 \tag{2.15}$$

and the 'constraint' equation for the vierbein

$$\dot{\phi}^2 = m^2 e^2 . \tag{2.16}$$

Hence the proper-time gauge corresponds to the choice $e = 1/m$ whereupon we recover the equation of motion (2.6) and the constraint (2.8).

It is interesting to note that the limit $m \to 0$ is singular in (2.1) but not in (2.14). This means that for a free massless and spinless particle we cannot eliminate the vierbein field to write an action with only $\phi^\mu(\tau)$.

3. The action for a relativistic massless spinning particle

The spinning particle is described by its position $\phi^\mu(\tau)$ together with an additional variable $\psi^\mu(\tau)$ which commutes with ϕ but anticommutes with itself,

$$\psi^\mu \psi^\nu + \psi^\nu \psi^\mu = 0 , \tag{3.1}$$

for any choice of μ and ν.

Proceeding as in sect. 2 we now construct a reparametrization invariant action by including the vierbein field e. The simplest Lagrangian which has this property is the most obvious generalization of (2.14) and is given by

$$L = \frac{1}{2}\left\{\frac{\dot{\phi}^2}{e} - i\psi \cdot \dot{\psi}\right\} . \tag{3.2}$$

This Lagrangian clearly transforms as a total derivative under reparametrizations if ψ transforms as a scalar like ϕ in (2.13). Explicitly, we find

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}\tau}(fL) , \tag{3.3}$$

so that the corresponding action is invariant. Because of the time component of the field ψ^μ there is a possibility that negative norm states may appear in the physical spectrum. In order to decouple them we require an additional invariance and, inspired by the Neveu-Schwarz-Ramond model, it seems natural to demand invariance under local supergauge transformations. This can be done by introducing a fermionic counter part, χ, to the vierbein field and writing the following Lagrangian

$$L = \frac{1}{2}\left(\frac{\dot{\phi}^2}{e} - i\psi\dot{\psi} - \frac{i}{e}\chi\dot{\phi} \cdot \psi\right) . \tag{3.4}$$

The action corresponding to (3.4) is now both reparametrization and supergauge invariant if χ transforms like e under reparametrizations

$$\delta\chi = f\dot{\chi} + \dot{f}\chi , \tag{3.5}$$

and if we choose the following supergauge transformations

$$\begin{aligned}
\delta\phi &= i\alpha\psi , \\
\delta\psi &= \alpha\left[\frac{\dot{\phi}}{e} - \frac{i}{2e}\chi\psi\right] , \\
\delta e &= i\alpha\chi , \\
\delta\chi &= 2\dot{\alpha} .
\end{aligned} \tag{3.6}$$

Here, α is an odd element of a Grassmann algebra which is an arbitrary function of τ. It is easy to check that under the transformations (3.6) the Lagrangian (3.4) is a total derivative

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{i}{2e}\alpha\dot{\phi}\cdot\psi\right) . \tag{3.7}$$

Thus our action has the required invariances. If we now interpret ϕ, ψ, e and χ as fields in one dimension we see that χ is the analogue of the Rarita-Schwinger field appearing in supergravity and the Lagrangian (3.4) represents the interaction between supergravity and supermatter in one dimension. This point of view was expanded in a separate report by Deser, Zumino and ourselves [7]. It is of interest to commute two supergauge transformations (3.6) on the fields and we find

$$\begin{aligned} [\delta_\beta, \delta_\alpha]\, e &= f\dot{e} + \dot{f}e + i\alpha'\chi\, , \\ [\delta_\beta, \delta_\alpha]\, \chi &= f\dot{\chi} + \dot{f}\chi + 2\dot{\alpha}'\, , \\ [\delta_\beta, \delta_\alpha]\, \phi &= f\dot{\phi} + i\alpha'\psi\, , \\ [\delta_\beta, \delta_\alpha]\, \psi &= f\dot{\psi} + \alpha'\left[\frac{\dot{\phi}}{e} - \frac{i}{2e}\chi\psi\right] , \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} f(\tau) &= \frac{2i\alpha\beta}{e}\, , \\ \alpha' &= -\tfrac{1}{2}f(\tau)\,\chi\, . \end{aligned} \tag{3.9}$$

Thus we find that the commutation of two supergauge transformations yields a reparametrization plus an additional supergauge transformation both of which are field dependent as in the case of pure supergravity [10]. This shows that there is no simple gauge group structure in the action, although the invariance is still enough to secure good physical properties as we shall see. The Euler-Lagrange equations obtained by varying (3.4) with respect to ϕ and ψ give us the equations of motion

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{2\dot{\phi}^\mu}{e} - i\chi\frac{\psi^\mu}{e}\right] = 0\, , \\ &2\dot{\psi}^\mu - \chi\frac{\dot{\phi}^\mu}{e} = 0\, , \end{aligned} \tag{3.10}$$

whilst variation with respect to e and χ gives us the constraint equations

$$\dot{\phi}^2 = \dot{\phi}\cdot\psi = 0\, . \tag{3.11}$$

4. Quantization in the proper-time gauge

Because of the invariances of the Lagrangian (3.4) we can choose the gauge in which $e = 1$ and $\chi = 0$. This corresponds to the proper-time gauge and the equations of motion become

$$\ddot{\phi}^\mu = \dot{\psi}^\mu = 0 \,, \tag{4.1}$$

with solutions

$$\begin{aligned} \phi^\mu &= q^\mu + p^\mu \tau \,, \\ \psi^\mu &= \text{constant Grassmann 4-vector} \,, \end{aligned} \tag{4.2}$$

whilst the constraints take the form

$$\dot{\phi}\psi = \dot{\phi}^2 = 0 \,. \tag{4.3}$$

Eq. (4.3) corresponds to the orthogonality of spin and velocity and the mass-shell condition. We may obtain (4.1) from the linearized Lagrangian

$$L = \tfrac{1}{2}[\dot{\phi}^2 - i\psi\dot{\psi}] \,, \tag{4.4}$$

which is now only invariant under translations in τ and constant supergauge transformations. To quantize the theory we should find the fundamental Poisson brackets and then pass to the (anti-) commutators in the usual way. In trying to carry through this programme we find that the defining equation for the momentum conjugate to ψ is itself a second-class constraint and we are obliged to use Dirac brackets. This problem has been studied extensively in ref. [5] and we find that the basic quantum relations are given by

$$\begin{aligned} [\hat{p}_\mu, \hat{q}_\nu] &= i g_{\mu\nu} \,, \\ [\psi_\mu, \psi_\nu]_+ &= g_{\mu\nu} \,. \end{aligned} \tag{4.5}$$

One solution to the anti-commutator relation is clearly given by

$$\psi_\mu = \sqrt{\tfrac{1}{2}}\gamma_\mu \,, \tag{4.6}$$

so that, in this case, the classical Grassmann algebra passes over to the Dirac-Clifford algebra in the quantum regime. The momentum eigenstates of the theory are then given by

$$|\psi\rangle = |p\rangle\, u(p) \,, \tag{4.7}$$

where

$$|p\rangle = e^{-ip\hat{q}}\,|0\rangle \,,$$

and p_μ is the eigenvalue of the momentum. $u(p)$ is a Dirac spinor. The first-class constraints (4.3) are imposed upon these states so we find

$$p^2|\psi_{\text{phys}}\rangle = \gamma\cdot p\,|\psi_{\text{phys}}\rangle = 0\,. \tag{4.8}$$

This realization of the commutation relations therefore describes a massless Dirac particle with the Dirac equation arising as a result of the constraints (4.3).

We can also satisfy the commutation relations (4.5) by setting

$$\psi_\mu = \tfrac{1}{2}\,[b_\mu^+ + b_\mu]\,, \tag{4.9}$$

with

$$\begin{aligned} &[b_\mu, b_\nu^+]_+ = g_{\mu\nu}\,,\\ &[b_\mu, b_\nu]_+ = [b_\mu^+, b_\nu^+]_+ = 0\,. \end{aligned} \tag{4.10}$$

The ground state is then given by

$$|0,p\rangle = \mathrm{e}^{-ip\cdot\hat{q}}\,|0\rangle\,, \tag{4.11}$$

which is a scalar state with momentum p and we can then generate the following further states

$$\begin{aligned} &b^{+\mu}|0,p\rangle\,, \qquad b^{+\mu}b^{+\nu}|0,p\rangle\,,\\ &b^{+\mu}b^{+\nu}b^{+\rho}|0,p\rangle\,, \qquad b^{+\mu}b^{+\nu}b^{+\rho}b^{+\sigma}|0,p\rangle\,. \end{aligned} \tag{4.12}$$

We now have to impose the constraints

$$\begin{aligned} &p^2|\psi_{\text{phys}}\rangle = 0\,,\\ &p\cdot b|\psi_{\text{phys}}\rangle = 0\,. \end{aligned} \tag{4.13}$$

These tell us that all the states are massless and transverse. This restricts the number of states to four as in the fermionic case and we get the following spectrum

$|0,p\rangle$ spin-0 particle ,

$b_i^+|0,p\rangle$ photon-like particle ($i = 1,2$) ,

and $(b_1^+b_2^+ - b_2^+b_1^+)|0,p\rangle$ antisymmetric tensor as in the closed string model [11].

Thus we see that our model has both a fermionic and a bosonic sector as in the case of the Neveu-Schwarz-Ramond string.

5. Interaction with external fields

To introduce external fields it is convenient to rewrite the linearized Lagrangian (4.4) in terms of superfields. We have

$$S = -\tfrac{1}{2}i\int \mathrm{d}\tau\,\mathrm{d}\theta\,\dot{y}^\mu D y_\mu\,, \tag{5.1}$$

where

$$y^\mu(\tau, \theta) = \phi^\mu(\tau) + i\theta\psi^\mu(\tau) , \tag{5.2}$$

$$Dy^\mu = \left(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial\tau}\right) y^\mu . \tag{5.3}$$

The constraints (4.3) may now be written in the form

$$\dot{y}^\mu Dy_\mu = 0 \tag{5.4}$$

whilst the equations of motion (4.2) are given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Dy^\mu = 0 . \tag{5.5}$$

We can now couple our 'superparticle' to electromagnetism in a U(1) gauge-invariant way by coupling the electric field minimally to the 'super co-ordinate' y^μ in the proper-time gauge. If we make the replacement $\dot{y}^\mu \to \dot{y}^\mu + eA^\mu$ in (5.1) we find that the current is given by

$$J^\mu(x) = -\tfrac{1}{2}ie\int \mathrm{d}\tau\,\mathrm{d}\theta\,\delta[x^\rho - y^\rho(\tau, \theta)]\,Dy^\mu , \tag{5.6}$$

and the interaction energy is

$$\int \mathrm{d}^4x\, J^\mu(x)\, A_\mu(x) . \tag{5.7}$$

We can perform the x and θ integrations to find the following interaction Lagrangian

$$L_{\mathrm{e.m.}} = \tfrac{1}{2}e[A^\mu(\phi)\,\dot{\phi}_\mu + \tfrac{1}{4}iF_{\mu\nu}[\psi^\mu, \psi^\nu]] . \tag{5.8}$$

Let us consider a plane-wave external field

$$A_\mu(\phi) = \epsilon_\mu \mathrm{e}^{-ik\cdot\phi} . \tag{5.9}$$

Then, passing to the quantum theory, we can define a vertex operator for the absorption of a photon of polarization ϵ_μ by

$$\hat{V}_\mu(K) = \tfrac{1}{4}e[\hat{p}_\mu, \mathrm{e}^{-ik\hat{\phi}}]_+ - \tfrac{1}{4}ie\sigma^{\mu\nu}k_\nu \mathrm{e}^{-ik\cdot\hat{\phi}} . \tag{5.10}$$

The operator $\hat{V}_\mu$ is also a matrix in spin space and the first term on the right-hand side is multiplied by the unit matrix. The vertex is given by

$$V_\mu(k) = \lim_{M\to 0}\frac{1}{M}\bar{u}(p')\,\langle p'|\hat{V}_\mu(k)|p\rangle\, u(p) , \tag{5.11}$$

where the factor $1/M$ has been introduced in order to get a finite result as in the case of the F_1 formalism of the Ramond model and of the O(2) symmetric dual model [12]. With the introduction of the mass we find that (5.11) can be conveniently rewritten with the help of the Gordon decomposition in the form

$$V_\mu(k) = \tfrac{1}{2} e\, u(p')\, \gamma_\mu\, u(p)\, \delta^{(4)}(p' - p - k)\,. \tag{5.12}$$

We can carry through a similar analysis for the case of an external gravitational field. We define the energy-momentum tensor for the particle by

$$T^{\mu\nu}(x) = -\tfrac{1}{2} i \int \mathrm{d}\tau\, \mathrm{d}\theta\, \delta(x - y(\tau, \theta))\, \dot{y}^\mu D y^\nu\,, \tag{5.13}$$

where symmetrization in the indices is to be understood. Let us consider a linearized external field which we may write in the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad h \text{ small}\,. \tag{5.14}$$

Then the interaction Lagrangian is given by

$$L_{\mathrm{grav}} = \tfrac{1}{2} h_{\mu\nu}(\phi)\, \dot{\phi}^\mu\, \dot{\phi}^\nu + \tfrac{1}{4} i \partial_\lambda h_{\mu\nu}(\phi)\, \{\dot{\phi}^\mu\, [\psi^\lambda, \psi^\nu] + \dot{\phi}^\nu [\psi^\lambda, \psi^\mu]\}\,, \tag{5.15}$$

where we have dropped terms that vanish because of the equations of motion. We again choose a plane-wave field

$$h_{\mu\nu} = \epsilon_{\mu\nu} \mathrm{e}^{-ik\cdot\phi}\,, \tag{5.16}$$

and the vertex operator is then given by

$$\begin{aligned} \hat{V}_{\mu\nu}(k) = \{ & [\hat{p}_\mu \hat{p}_\nu, \mathrm{e}^{-ik\hat{\phi}}]_+ + \hat{p}_\mu \mathrm{e}^{-ik\cdot\hat{\phi}}\, \hat{p}_\nu + \hat{p}_\nu\, \mathrm{e}^{-ik\hat{\phi}}\, \hat{p}_\mu \\ & + \tfrac{1}{4} i\, k^\lambda \sigma_{\nu\lambda} [\hat{p}_\mu, \mathrm{e}^{-ik\hat{\phi}}]_+ + \tfrac{1}{4} i k^\lambda \sigma_{\mu\lambda} [\hat{p}_\nu, \mathrm{e}^{-ik\hat{\phi}}]_+ \}\,. \end{aligned} \tag{5.17}$$

Defining the vertex by a similar limiting procedure as in the electromagnetic case, we find

$$\begin{aligned} V_{\mu\nu}(k) = & \lim_{M\to 0} \frac{1}{2M}\, \bar{u}(p')\, \{4P_\mu P_\nu + ik^\lambda \sigma_{\mu\lambda} P_\nu + ik^\lambda \sigma_{\nu\lambda} P_\mu\} \\ & \times u(p)\, \delta^{(4)}(p' - p - k)\,, \end{aligned} \tag{5.18}$$

where

$$P_\mu = \tfrac{1}{2}(p'_\mu + p_\mu)\,.$$

This may be rearranged as in the electromagnetic case and we finally obtain

$$V_{\mu\nu}(k) = \bar{u}(p')\, [\gamma_\mu P_\nu + \gamma_\nu P_\mu]\, u(p)\, \delta^{(4)}(p' - p - k)\,. \tag{5.19}$$

We thus conclude that the formalism reproduces the conventional results as required.

6. The action for a massive spinning particle

The action for the massive case has been treated by Berezin and Marinov [9] who deduced a Lagrange multiplier form for it. They discussed local supersymmetry but did not interpret their Lagrange multipliers as a vierbein field and its fermionic counterpart, nor did they discuss variations of these fields. In constructing the massive case, the key point to notice is that the previous case is chirally invariant. At first sight it seems difficult to change the constraint $\dot{\phi} \cdot \psi = 0$ to include a mass since the constraint is fermionic. Because of the chiral invariance we can, however, let $\psi_\mu \to \gamma_5 \gamma_\mu$ in the quantization procedure. Thus if we introduce an additional (Minkowski scalar) Grassmann variable ψ_5, which goes over to γ_5 in the quantization procedure, this field could carry a mass in the constraint. We further have to introduce a 'cosmological' term $\frac{1}{2}em^2$ in the Lagrangian to give the mass shell condition. With this insight we can in fact find the new piece to add to the Lagrangian (3.4) which transforms as a total derivative under supergauge transformations. This new piece is

$$L_5 = \tfrac{1}{2}em^2 + \tfrac{1}{2}i\psi_5\dot{\psi}_5 - \tfrac{1}{2}im\psi_5\chi \,, \tag{6.1}$$

where

$$\delta\psi_5 = m\alpha(\tau) + \frac{i}{me}\alpha\psi_5(\dot{\psi}_5 - \tfrac{1}{2}m\chi) \tag{6.2}$$

under local supergauge transformations. Thus the total action in the massive case is

$$S = \tfrac{1}{2}\int \mathrm{d}\tau \left\{ \frac{\dot{\phi}^2}{e} + em^2 - i(\psi\dot{\psi} - \psi_5\dot{\psi}_5) - i\chi\left(\frac{\psi\cdot\dot{\phi}}{e} - m\psi_5\right)\right\} \,. \tag{6.3}$$

From the equations of motion we can in fact solve for χ in terms of ψ_5 and thus restore the balance between the number of Bose and Fermi fields appearing in the theory. However, this gives dynamics to the 'gravitational' part (e, ψ_5) and so we retain ψ_5 as an independent variable. The Euler-Lagrange equations following from (6.3) are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{2\dot{\phi}}{e} - \frac{i\chi\psi}{e}\right] = 0 \,,$$

$$\dot{\psi} - \frac{\chi\dot{\phi}}{2e} = 0 \,, \tag{6.4}$$

$$2\dot{\psi}_5 - m\chi = 0 \,,$$

$$\frac{\dot{\phi}\psi}{e} - m\psi_5 = 0 \,,$$

$$\frac{-\dot{\phi}^2}{e^2} + m^2 + \frac{i\chi\dot{\phi}\psi}{e^2} = 0 \,. \tag{6.5}$$

The equations (6.4) can be considered to be the equations of motion and the equations (6.5) are the constraints. As in the massless case we are allowed to choose two gauge conditions because of the invariance and we set $e = 1/m$, $\chi = 0$. In this, the proper-time, gauge the equations of motion are

$$\ddot{\phi}^{\mu} = \dot{\psi}^{\mu} = \dot{\psi}^{5} = 0\,, \tag{6.6}$$

while the constraints become

$$\begin{aligned} &\dot{\phi}\psi - \psi_5 = 0\,, \\ &\dot{\phi}^2 - 1 = 0\,. \end{aligned} \tag{6.7}$$

We now pass to the fermionic quantization of the system. In this gauge we have

$$p_{\psi} = \tfrac{1}{2}i\psi\,, \qquad p_{\psi_5} = -\tfrac{1}{2}i\psi_5\,, \qquad p_{\phi} = \hat{p} = m\dot{\phi}\,. \tag{6.8}$$

As in the massless case we have to be careful because of the appearance of second-class constraints. The correct commutation relations are found to be

$$\begin{aligned} &[\hat{p}_{\mu}, \hat{q}_{\nu}] = ig_{\mu\nu}\,, \\ &[\psi_{\mu}, \psi_{\nu}]_{+} = g_{\mu\nu}\,, \\ &[\psi_5, \psi_5]_{+} = -1\,. \end{aligned} \tag{6.9}$$

A solution to these equations is given by choosing

$$\psi_{\mu} = \sqrt{\tfrac{1}{2}}\gamma_5\gamma_{\mu}\,, \qquad \psi_5 = \sqrt{\tfrac{1}{2}}\,\gamma_5\,. \tag{6.10}$$

We see once again the Dirac-Clifford algebra emerge from the classical Grassmann algebra.

Of course, γ_5 is not independent of the γ_{μ}'s but the theory is fully consistent. The momentum eigenstates are given by

$$|\psi\rangle = e^{-ip\cdot\hat{q}}\,|0\rangle\, u(p)\,, \tag{6.11}$$

and we further have to impose the constraints (6.7) upon physical states. We obtain

$$(p^2 - m^2)\,|\psi_{\text{phys}}\rangle = \gamma_5(\not{p} - m)|\psi_{\text{phys}}\rangle = 0\,. \tag{6.12}$$

The last condition may be multiplied by γ_5 to recover the massive Dirac equation.

We conclude this section showing that as in the case of the spinless particle the "gravitational" fields e and χ can be eliminated from the Lagrangian (6.3) by using the equations of motion for e and χ.

The equation of motion for e is given by

$$m^2e^2 = (\dot{\phi} - \tfrac{1}{2}i\chi\psi)^2\,, \tag{6.13}$$

where the relation $(\chi\psi)^2 = 0$ has been used. The equation of motion for ψ_5 is

$$2\dot{\psi}_5 = m\chi \,. \tag{6.14}$$

Substituting those equations in (6.3) we get the Lagrangian for a spinning particle expressed only in terms of the variables ϕ, ψ and ψ_5,

$$L = m\sqrt{\left(\dot{\phi} - \frac{i}{m}\dot{\psi}_5\,\psi\right)^2} - \tfrac{1}{2}i\psi\dot{\psi} - \tfrac{1}{2}i\psi_5\dot{\psi}_5 \;. \tag{6.15}$$

It is easy to check that (6.15) transforms as a total derivative under the local super-gauge transformations

$$\delta\phi = i\alpha\psi \,, \qquad \delta\psi = \alpha\pi \,, \tag{6.16}$$

$$\delta\psi_5 = m\alpha \,,$$

where

$$\pi^\mu = \frac{\partial L}{\partial\dot{\phi}_\mu} \,. \tag{6.17}$$

One gets in fact

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\tfrac{1}{2}i\alpha(\pi\psi + m\psi_5)\right] . \tag{6.18}$$

7. Inclusion of internal symmetry

In this section we extend our previous results to a system containing an internal O(2) symmetry. Although this may be done in an analogous manner to the preceeding case, it is more elegant to formulate the theory first of all in superspace. In a previous publication [7] we have shown that the massless spinning particle may be described in a two-dimensional superspace with co-ordinates $z^M = (\tau, \theta)$, and that the theory presented here corresponds to the choice of a special gauge. We may repeat the arguments given in ref. [7] to derive corresponding results for the O(2) case. The appropriate superspace is now three-dimensional with co-ordinates $z^M = (\tau, \theta^1, \theta^2)$ where τ is the ordinary bosonic co-ordinate and $\theta^m (m = 1,2)$ are two Grassmann co-ordinates. Our basic field variables are a scalar superfield $X(z)$, which is also a Minkowski four-vector, and a super-vierbein field $E_M^A(z)$ *. In addition, we also have the inverse vierbein E_A^M satisfying

$$E_B^M\, E_M^A = \delta_B^A \,. \tag{7.1}$$

* Throughout this section $M(A)$ is a curved (tangent-space) index: $M = (\mu, m)$; $A = (\alpha, a)$ where $\mu(\alpha)$ are bosonic indices taking on only one value and $m(a)$ are fermionic indices taking on two values each. We also suppress the Lorentz index on X.

The action for the system is given by

$$S = \frac{1}{4}\int d^3z\, E\, \epsilon^{ab} E_a^M \partial_M X \cdot E_b^N \partial_N X\,, \tag{7.2}$$

where E is the (generalized) determinant of E_M^A. This action is invariant under general co-ordinate transformations in superspace,

$$\delta E_M^A = \xi^N \partial_N E_M^A + \partial_M \xi^N E_N^A\,,$$

$$\delta X = \xi^M \partial_M X\,, \tag{7.3}$$

and also under tangent space rotations,

$$\delta E_M^a = -E_M^\alpha \varphi^a + E_M^b\, \epsilon_b^a T\,,$$

$$\delta E_M^\alpha = \delta X = 0\,. \tag{7.4}$$

Here the φ^a's are fermionic parameters and T is a bosonic parameter corresponding to local O(2) rotations. At this stage all the parameter ξ^M, φ^a, T are arbitrary functions of z, but we may limit the invariance to be of the desired "supergravity" form by choosing the gauge in which

$$E_M^\alpha = \Lambda \bar{E}_M^\alpha\,, \qquad E_M^a = \Lambda^{1/2} \bar{E}_M^a\,, \tag{7.5}$$

where $\bar{E}_M^A$ is the flat vierbein given by

$$\bar{E}_\mu^\alpha = 1\,, \quad \bar{E}_\mu^a = 0\,,$$

$$\bar{E}_m^\alpha = -i\theta_m\,, \qquad \bar{E}_M^a = \delta_m^a\,, \tag{7.6}$$

and Λ is a scalar superfield. In order to stay in this gauge we find that the parameters must take on the following form:

$$\xi^\mu = a + i\theta^m \beta^m\,,$$

$$\xi^m = \beta^m + \tfrac{1}{2}\theta^m \dot{a} + \epsilon^{mn}\theta^n t + i\theta^m \theta^n \dot{\beta}^n\,,$$

$$T = t - i\theta^m \epsilon^{mn} \dot{\beta}^n - \tfrac{1}{4} i \epsilon^{mn} \theta^m \theta^n \ddot{a}\,,$$

$$\varphi^a = \Lambda^{-1/2} \dot{\xi}^a\,. \tag{7.7}$$

The three remaining independent parameters a, β^m and t are arbitrary functions of τ and correspond to general co-ordinate transformations in τ-space, local supergauge transformations and local O(2) rotations, respectively. The Λ-field transform as follows:

$$\delta\Lambda = \xi^M \partial_M \Lambda + \dot{\xi}^\mu \Lambda + i\theta^m \dot{\xi}^m \Lambda\,, \tag{7.8}$$

so that, expanding the superfields,

$$\Lambda = e + i\theta^m \chi'^m + \tfrac{1}{2} i\epsilon^{mn}\theta^m\theta^n f' ,$$

$$X = \phi + i\theta^m \psi'^m + \tfrac{1}{2} i\epsilon^{mn}\theta^m\theta^n F' . \tag{7.9}$$

We find for the component fields,

$$\delta e = a\dot{e} + \dot{a}e + i\beta^m \chi'^m ,$$

$$\delta\chi'^m = a\dot{\chi}'^m + \tfrac{3}{2}\dot{a}\chi'^m + \beta^m \dot{e} + 2\dot{\beta}^m e + \epsilon^{mn}\beta^n f' - t\epsilon^{mn}\chi'^n ,$$

$$\delta f' = af' + 2\dot{a}f' - i\epsilon^{mn}\beta^m\dot{\chi}'^n - 3i\epsilon^{mn}\dot{\beta}^m \chi'^n + 2\dot{t}e , \tag{7.10}$$

and

$$\delta\phi = a\dot{\phi} + i\beta^m \psi'^m ,$$

$$\delta\psi'^m = a\dot{\psi}'^m + \tfrac{1}{2}\dot{a}\psi'^m + \beta^m\dot{\phi} + \epsilon^{mn}\beta^n F' - t\epsilon^{mn}\psi'^n ,$$

$$\delta F' = a\dot{F}' + \dot{a}F' - i\epsilon^{mn}(\dot{\beta}^m\psi' + \beta^m\dot{\psi}'). \tag{7.11}$$

We may redefine the fields such that e, χ^m and f transform as 'co-vectors' under reparametrizations whilst ϕ, ψ^m and F transform as 'scalars'. This is accomplished by setting

$$\chi'^m = \sqrt{e}\chi^m , \qquad f' = ef ,$$

$$\psi'^m = \sqrt{e}\psi^m , \qquad F' = eF , \tag{7.12}$$

so that under a-transformations,

$$\delta e, \chi^m, f = a(\dot{\ }) + \dot{a}(\,) ,$$

$$\delta\phi, \psi, F = a(\dot{\ }) , \tag{7.13}$$

as desired. If we further set $\beta^m = e^{-1/2}\alpha^m$ we find the following supergauge transformations for the unprimed fields:

$$\delta e = i\alpha^m \chi^m ,$$

$$\delta\chi^m = 2\dot{\alpha}^m + \epsilon^{mn}\alpha^n f - \frac{i}{2e}\alpha^n\chi^n\chi^m ,$$

$$\delta f = -\frac{i}{e}\left[\epsilon^{mn}\alpha^m\dot{\chi}^n + 3\epsilon^{mn}\dot{\alpha}^m\chi^n - \epsilon^{mn}\alpha^m\chi^n\frac{\dot{e}}{e} + \alpha^n\chi^n f\right] , \tag{7.14}$$

$$\delta\phi = i\alpha^m\psi^m ,$$

$$\delta\psi^m = \frac{\alpha^m}{e}\ \dot{\phi} + \epsilon^{mn}\alpha^n F - \frac{i}{2e}\alpha^n\chi^n\psi^m\ ,$$

$$\delta F = -\frac{i}{e}\{\epsilon^{mn}[\dot{\alpha}^m\psi^m + \alpha^m\dot{\psi}^m] + \alpha^n\chi^n F\}\ , \tag{7.15}$$

whilst for local O(2) transformations we have

$$\delta e = 0\ ,$$

$$\delta\chi^m = -t\epsilon^{mn}\chi^n\ ,$$

$$\delta f = 2\dot{t}\ , \tag{7.16}$$

$$\delta\phi = \delta F = 0\ ,$$

$$\delta\psi^m = -t\epsilon^{mn}\psi^n\ . \tag{7.17}$$

It is interesting to compute the effect of commuting two supergauge transformations. We find that for all the above fields we obtain co-ordinate transformations, supergauge transformations and local O(2) rotations. Explicitly, for any field A we have

$$[\delta_\beta, \delta_\alpha]\ A = \delta_a A + \delta_\gamma A + \delta_t A\ , \tag{7.18}$$

where

$$a = \frac{2i}{e}\ \alpha^m\beta^m\ ,$$

$$\gamma^m = -\frac{i}{e}\epsilon^{mn}\alpha^m\beta^n\ ,$$

$$t = -\frac{i}{e}\epsilon^{mn}(\alpha^m\dot{\beta}^n - \beta^m\dot{\alpha}^n)\ . \tag{7.19}$$

Thus the algebra closes in that the commutation leads to field-dependent transformations.

In this gauge the action (7.2) becomes

$$S = \frac{1}{4}\int d^3z\ \epsilon^{mn}D_m X D_n X\Lambda^{-1}\ , \tag{7.20}$$

where

$$D_m = \frac{\partial}{\partial\theta^m} + i\theta_m\frac{\partial}{\partial\tau}\ .$$

Expanding the superfields and integrating over θ we find the τ-space Lagrangian

$$L = \frac{1}{2}\left\{\frac{1}{e}\dot{\phi}^2 + i\dot{\psi}^m\psi^m + eF^2 + \frac{i}{e}\dot{\phi}\cdot\psi^m\chi^m\right.$$
$$\left. -\tfrac{1}{2}i\epsilon^{mn}\psi^m\psi^n f - \frac{1}{e}\psi^m\chi^m\psi^n\chi^n + i\epsilon^{mn}\psi^m\chi^n F\right\}\ . \tag{7.21}$$

The corresponding τ-space action is manifestly coordinate and O(2) invariant and using the transformations (7.14), (7.15), we find that the Lagrangian transforms as a total derivative under supergauge transformations,

$$\delta L = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{i}{e}\alpha^m\psi^m\dot{\phi} - \frac{1}{e}\alpha^m\psi^m\psi^n\chi^n - i\epsilon^{mn}\alpha^m\psi^n F\right\} . \tag{7.22}$$

The Euler-Lagrange equations derived from (7.21) are given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{2\dot{\phi}}{e} + i\frac{\psi^m\chi^m}{e}\right] = 0 ,\\
&2\dot{\psi}^m - \frac{1}{e}\dot{\phi}\chi^m + \epsilon^{mn}\psi^n f - \frac{2i}{e}\chi^m\psi^n\chi^n - \epsilon^{mn}\chi^n F = 0 ,\\
&eF + i\epsilon^{mn}\psi^m\chi^n = 0 ,
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
&\dot{\phi}^2 + i\dot{\phi}\psi^m\chi^m + (\psi^m\chi^m)^2 = 0 ,\\
&\epsilon^{mn}\psi^m\psi^n = 0 ,\\
&\dot{\phi}\psi^m + 2i\psi^m\psi^n\chi^n + \epsilon^{mn}\psi^n F = 0 .
\end{aligned} \tag{7.24}$$

Eqs. (7.23) may be considered as the equations of motion whilst eqs. (7.24) are the constraints. These may be simplified considerably by choosing the proper-time gauge ($\Lambda = 1$) whereupon the equations of motion become

$$\ddot{\phi} = \dot{\psi}^m = F = 0 , \tag{7.25}$$

and the constraints become

$$\dot{\phi}^2 = \dot{\phi}\psi^m = \epsilon^{mn}\psi^m\psi^n = 0 . \tag{7.26}$$

The first two of these last equations are straightforward generalizations of the case of only one θ given in sect. 3, while the third condition will give us colour confinement in the quantum theory.

The solutions to (7.25) are clearly given by

$$\begin{aligned}
&\phi^\mu = q^\mu + p^\mu\tau ,\\
&\psi^m_\mu = C^m_\mu ,\\
&F = 0 ,
\end{aligned} \tag{7.27}$$

where μ is now a Lorentz index, and the quantum theory is defined by giving the basic commutation relations

$$\begin{aligned}
&[q_\mu, q_\nu] = ig_{\mu\nu} ,\\
&[C^m_\mu, C^n_\nu]_+ = g_{\mu\nu}\delta^{mn} ,
\end{aligned} \tag{7.28}$$

together with imposing the constraints (7.26) on the physical states. The anti-commutation relation (7.28) may be realized by setting

$$C_m^\mu = \tfrac{1}{2}(b_m^\mu + b_m^{+\mu}) , \tag{7.29}$$

where

$$[b_m^\mu, b_n^{+\nu}]_+ = g^{\mu\nu}\delta_{mn} . \tag{7.30}$$

The constraints then become

$$\hat{p}^2|\psi_{\text{phys}}\rangle = 0 ,$$

$$\epsilon^{mn}b^{+m}b^n|\psi_{\text{phys}}\rangle = 0 ,$$

$$\hat{p}\cdot b^m|\psi_{\text{phys}}\rangle = 0 . \tag{7.31}$$

The operator $\epsilon^{mn}b^{+m}b^n$ generates global O(2) transformations, so that the second of the conditions (7.31) implies that the physical states must be O(2) singlets and that we have colour confinement. Defining momentum eigenstates by

$$|0, p\rangle = \mathrm{e}^{-ip\cdot\hat{q}}|0\rangle , \tag{7.32}$$

we then have the following spectrum of physical states:

$$|0,p\rangle, \quad \delta^{mn}b_\perp^{+m}\, b_\perp^{+n}\;|0,p\rangle , \qquad \epsilon^{mn}b_\perp^{+m}b_\perp^{+n}\;|0,p\rangle ,$$

$$b_1^{1+}b_2^{1+}b_1^{2+}b_2^{+2}|0,p\rangle , \tag{7.34}$$

where $b_\perp = (b_1, b_2)$ if we choose $p = (p_3, 0, 0, p_3)$.

The system therefore turns out to describe six massless states corresponding to four spin-0 particles and one spin-2 particle.

After the completion of this work we received two preprints by P.A. Collins and R.W. Tucker (Lancaster) and A. Barducci, R. Casalbuoni, and L. Lusanna (Firenze) in which the case of the massive spinning particle also is discussed.

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References

[1] M. Jacob, Dual theory (North-Holland, Amsterdam);
P. Frampton, Dual models (Wiley, New York).

[2] A. Neveu and J. Schwarz, Nucl. Phys. B80 (1971) 499;
P. Ramond, Phys. Rev. D3 (1971) 2415.
[3] B. Zumino, Supersymmetry, CERN preprint TH 2120 (1976).
[4] F.A. Berezin and M.S. Marinov, JETP Letters (Sov.Phys.) 21 (1975) 320.
[5] R. Casalbuoni, Phys. Letters 62B (1976) 49; CERN preprint TH 2139 (1976).
[6] P.A. Dirac, Lectures at Yeshiva University, 1964 p. 47.
[7] L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. Howe, Phys. Letters 64B (1976) 435.
[8] S. Deser and B. Zumino, Phys. Letters 62B (1976) 335;
D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D13 (1976) 3214.
[9] F.A. Berezin and M.S. Marinov, ITEP-43 preprint (1976).
[10] D.Z. Freedman and P. van Nieuwenhuizen, Properties of supergravity theory, ITP-SB-76-25 (1976).
[11] E. Cremmer and J. Scherk, Nucl. Phys. B72 (1974) 117;
M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273.
[12] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, G. Gliozzi, R. Musto, R. Pettorino and J. Schwarz, Nucl. Phys. B111 (1976) 77.