the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires that we evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find a particular solution of

$$y'' + 4y = 3\csc t. \tag{1}$$

Observe that this problem does not fall within the scope of the method of undetermined coefficients because the nonhomogeneous term $g(t) = 3\csc t$ involves a quotient (rather than a sum or a product) of $\sin t$ or $\cos t$. Therefore, we need a different approach. Observe also that the homogeneous equation corresponding to Eq. (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of Eq. (2) is

$$y_c(t) = c_1 \cos 2t + c_2 \sin 2t. \tag{3}$$

The basic idea in the method of variation of parameters is to replace the constants c_1 and c_2 in Eq. (3) by functions $u_1(t)$ and $u_2(t)$, respectively, and then to determine these functions so that the resulting expression

$$y = u_1(t)\cos 2t + u_2(t)\sin 2t \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2 we need to substitute for y from Eq. (4) in Eq. (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and u_2. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.

Returning now to Eq. (4), we differentiate it and rearrange the terms, thereby obtaining

$$y' = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t. \tag{5}$$

Keeping in mind the possibility of choosing a second condition on u_1 and u_2, let us require the last two terms on the right side of Eq. (5) to be zero; that is, we require that

$$\boxed{u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0.} \tag{6}$$

It then follows from Eq. (5) that

$$y' = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t. \tag{7}$$

Although the ultimate effect of the condition (6) is not yet clear, at the very least it has simplified the expression for y'. Further, by differentiating Eq. (7), we obtain

$$y'' = -4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t. \tag{8}$$

Then, substituting for y and y'' in Eq. (1) from Eqs. (4) and (8), respectively, we find that u_1 and u_2 must satisfy

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t. \tag{9}$$

Summarizing our results to this point, we want to choose u_1 and u_2 so as to satisfy Eqs. (6) and (9). These equations can be viewed as a pair of linear *algebraic* equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$. Equations (6) and (9) can be solved in various ways. For example, solving Eq. (6) for $u_2'(t)$, we have

$$u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}. \tag{10}$$

Then, substituting for $u_2'(t)$ in Eq. (9) and simplifying, we obtain

$$u_1'(t) = -\frac{3\csc t \sin 2t}{2} = -3\cos t. \tag{11}$$

Further, putting this expression for $u_1'(t)$ back in Eq. (10) and using the double angle formulas, we find that

$$u_2'(t) = \frac{3\cos t\cos 2t}{\sin 2t} = \frac{3(1 - 2\sin^2 t)}{2\sin t} = \frac{3}{2}\csc t - 3\sin t. \tag{12}$$

Having obtained $u_1'(t)$ and $u_2'(t)$, the next step is to integrate so as to obtain $u_1(t)$ and $u_2(t)$. The result is

$$u_1(t) = -3\sin t + c_1 \tag{13}$$

and

$$u_2(t) = \tfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2. \tag{14}$$

Finally, on substituting these expressions in Eq. (4), we have

$$\begin{aligned} y = {} & -3\sin t\cos 2t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t + 3\cos t\sin 2t \\ & + c_1\cos 2t + c_2\sin 2t, \end{aligned}$$

or

$$y = 3\sin t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t + c_1\cos 2t + c_2\sin 2t. \tag{15}$$

The terms in Eq. (15) involving the arbitrary constants c_1 and c_2 are the general solution of the corresponding homogeneous equation, while the remaining terms are a particular solution of the nonhomogeneous equation (1). Therefore Eq. (15) is the general solution of Eq. (1).

In the preceding example the method of variation of parameters worked well in determining a particular solution, and hence the general solution, of Eq. (1). The next question is whether this method can be applied effectively to an arbitrary equation. Therefore we consider

$$y'' + p(t)y' + q(t)y = g(t), \tag{16}$$

where p, q, and g are given continuous functions. As a starting point, we assume that we know the general solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{17}$$

of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{18}$$

This is a major assumption because so far we have shown how to solve Eq. (18) only if it has constant coefficients. If Eq. (18) has coefficients that depend on t, then usually the methods described in Chapter 5 must be used to obtain $y_c(t)$.

The crucial idea, as illustrated in Example 1, is to replace the constants c_1 and c_2 in Eq. (17) by functions $u_1(t)$ and $u_2(t)$, respectively; this gives

$$y = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{19}$$

Then we try to determine $u_1(t)$ and $u_2(t)$ so that the expression in Eq. (19) is a solution of the nonhomogeneous equation (16) rather than the homogeneous equation (18). Thus we differentiate Eq. (19), obtaining

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t). \tag{20}$$

As in Example 1, we now set the terms involving $u_1'(t)$ and $u_2'(t)$ in Eq. (20) equal to zero; that is, we require that

$$\boxed{u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.} \tag{21}$$

Then, from Eq. (20), we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{22}$$

Further, by differentiating again, we obtain

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t). \tag{23}$$

Now we substitute for y, y', and y'' in Eq. (16) from Eqs. (19), (22), and (23), respectively. After rearranging the terms in the resulting equation we find that

$$\begin{aligned} u_1(t)&[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ &+ u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ &\qquad + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \end{aligned} \tag{24}$$

Each of the expressions in square brackets in Eq. (24) is zero because both y_1 and y_2 are solutions of the homogeneous equation (18). Therefore Eq. (24) reduces to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{25}$$

Equations (21) and (25) form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. They correspond exactly to Eqs. (6) and (9) in Example 1.

By solving the system (21), (25) we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}, \tag{26}$$

where $W(y_1,y_2)$ is the Wronskian of y_1 and y_2. Note that division by W is permissible since y_1 and y_2 are a fundamental set of solutions, and therefore their Wronskian is nonzero. By integrating Eqs. (26) we find the desired functions $u_1(t)$ and $u_2(t)$, namely,

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_2. \tag{27}$$

Finally, substituting from Eq. (27) in Eq. (19) gives the general solution of Eq. (16). We state the result as a theorem.

Theorem 3.7.1 If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation (18) corresponding to the nonhomogeneous equation (16),

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (16) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt, \tag{28}$$

and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{29}$$

as prescribed by Theorem 3.6.2.

By examining the expression (28) and reviewing the process by which we derived it, we can see that there may be two major difficulties in using the method of variation of parameters. As we have mentioned earlier, one is the determination of $y_1(t)$ and $y_2(t)$, a fundamental set of solutions of the homogeneous equation (18), when the coefficients in that equation are not constants. The other possible difficulty is in the evaluation of the integrals appearing in Eq. (28). This depends entirely on the nature of the functions y_1, y_2, and g. In using Eq. (28), be sure that the differential equation is exactly in the form (16); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that Eq. (28) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions.

PROBLEMS

In each of Problems 1 through 4 use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$