The probability for multinomial distribution is given by, ${p_{11}}^{x_{11}}{p_{12}}^{x_{12}}{p_{21}}^{x_{21}}(1 - p_{11} - p_{12} - p_{21} - p_{22})^{x_{22}}$. Let $p_{22} = 1 - p_{11} - p_{12} - p_{21} - p_{22}$ we then get the log-likelihood,

$$\ln\left[L(p_{ij}; x)\right] = \sum_i \sum_j x_{ij} \ln p_{ij}$$

The multinomial coefficient is omitted since it will be eliminated upon partial differentiating and equating it to 0. Let $\ln\left[L(p_{ij}; x)\right] = L^*\left(p_{ij}; x\right)$. Since $L(p_{ij}; x)$ is subject to $\sum p_{ij} - 1 = 0$ (let $H\left(p_{ij}\right) = \sum p_{ij} - 1$), we therefore introduce a single λ Lagrange multiplier and compute the partial derivatives of

$$\frac{\partial L^*(p_{ij};x)}{\partial p_{ij}} \text{ and } \frac{\partial H(p_{ij})}{\partial p_{ij}} \quad i, j = 1,2$$

$$\frac{\partial L^*(p_{ij};x)}{\partial p_{ij}} = \frac{\partial}{\partial p_{ij}}\left(\sum_i \sum_j x_{ij} \ln p_{ij}\right)$$

$$= \frac{x_{ij}}{p_{ij}}$$

$$\frac{\partial H(p_{ij})}{\partial p_{ij}} = \frac{\partial}{\partial p_{ij}}\left(\sum p_{ij} - 1\right) = 1$$

Solve the all the equations for $i, j = 1,2$.

$$\frac{\partial L^*(p_{ij};x)}{\partial p_{ij}} = \lambda \frac{\partial H(p_{ij})}{\partial p_{ij}} \rightarrow \frac{x_{ij}}{p_{ij}} = \lambda$$

i.e.

$$p_{11} = \frac{x_{11}}{\lambda}, p_{12} = \frac{x_{12}}{\lambda}, p_{21} = \frac{x_{21}}{\lambda}, p_{22} = \frac{x_{22}}{\lambda}$$

Substituting all these equations to $H\left(p_{ij}\right)$ will give us,

$$H\left(p_{ij}\right) = \sum p_{ij} - 1 = 0$$

$$= p_{11} + p_{12} + p_{21} + p_{22} - 1 = 0$$

$$\rightarrow \quad 1 = \frac{x_{11}}{\lambda} + \frac{x_{12}}{\lambda} + \frac{x_{21}}{\lambda} + \frac{x_{22}}{\lambda}$$

$$\lambda = \sum_j \sum_i x_{ij}$$

Thus, the maximum likelihood estimators are

$$\widehat{p_{11}} = \frac{x_{11}}{\sum_j \sum_i x_{ij}}, \widehat{p_{12}} = \frac{x_{12}}{\sum_j \sum_i x_{ij}}, \widehat{p_{21}} = \frac{x_{21}}{\sum_j \sum_i x_{ij}},$$

$$\widehat{p_{22}} = \frac{x_{22}}{\sum_j \sum_i x_{ij}}$$

Since $x_{ij}'s$ are nonnegative integers that add to n, i.e.,

$\sum_j \sum_i x_{ij} = n$, then

$$\widehat{p_{11}} = \frac{x_{11}}{n}, \widehat{p_{12}} = \frac{x_{12}}{n}, \widehat{p_{21}} = \frac{x_{21}}{n}, \widehat{p_{22}} = \frac{x_{22}}{n}$$

But $\sum N_{ij} = n$,

$$n = N_{11} + N_{12} + N_{21} + N_{22}$$

$$= 90 + 6 + 3 + 1$$

$$= 100$$

Therefore, the final MLE for our parameters are,

$$\widehat{p_{11}} = \frac{x_{11}}{100}, \widehat{p_{12}} = \frac{x_{12}}{100}, \widehat{p_{21}} = \frac{x_{21}}{100}, \widehat{p_{22}} = \frac{x_{22}}{100}$$