



A coupled non-linear mathematical model of parametric resonance of ships in head seas

Marcelo A.S. Neves ^{*}, Claudio A. Rodríguez

Department of Naval Architecture and Ocean Engineering, LabOceano/COPPE, Federal University of Rio de Janeiro, C.P. 68.508, Rio de Janeiro, CEP 21945-970, Brazil

ARTICLE INFO

Article history:

Received 3 March 2007

Received in revised form 31 July 2008

Accepted 6 August 2008

Available online 15 August 2008

Keywords:

Ship stability

Roll motion

Parametric resonance

Hill equation

ABSTRACT

The present paper describes a non-linear third order coupled mathematical model of parametric resonance of ships in head seas. Coupling is contemplated by considering the restoring modes of heave, roll and pitch motions. Numerical simulations employing this new model are compared to experimental results corresponding to excessive motions of a transom stern fishing vessel in head seas. It is shown that this enhanced model matches its results with the experiments more closely than a second order model. It is shown that the new model, due to the introduction of the third order terms, entails qualitative differences when compared to the more commonly used second order model. The variational equation of the roll motion will not be in the form of a Mathieu equation. In fact, it is shown in the paper that the associated time-dependent equation falls into the category of a Hill equation. Additionally, a hardening effect is analytically derived, related to the third order coupling of modes and wave passage effects.

Limits of stability corresponding to the linear variational equation of the coupled roll motion are analytically derived. Numerical limits of stability corresponding to the non-linear equations are computed and compared to the analytically derived limits.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

The phenomenon of parametric rolling of ships has been recognized by naval architects since the late 1930s. It is noted that in pure head or following seas the transverse symmetry of the ship would imply that no wave-induced roll exciting moment should be present. Nevertheless, for certain frequencies of wave encounter, it is found that a small initial disturbance in roll can trigger an oscillatory rolling that can grow to appreciable amplitude after only a few cycles [1–5]. Much of such attention has been devoted to the particular configuration of longitudinal regular waves, either with or without speed, bow or stern waves. For many years, more attention has been given to parametric rolling in astern seas [4,6–8]. Roll motion of ships has usually been modelled as an uncoupled Mathieu type equation [9]. Considering the well-known existence of the Mathieu resonant frequencies, focus has been concentrated on the first region of instability, defined by the proximity of encounter frequency to twice the roll natural frequency.

More recently parametric excitation in head seas has received wide attention due to some recorded accidents. France et al. [10] reported on strong roll amplification in the case of a post-Panamax container ship, which suffered severe damage to containers and structure caused by excessive accelerations in roll. Dallinga et al. [11], Luth and Dallinga [12] reported on the development of head seas parametric resonance in cruise vessels. Levadou and Palazzi [13] attempted to evaluate the

^{*} Corresponding author. Tel.: +55 21 3867 6768; fax: +55 21 2662 8716.

E-mail address: masn@peno.coppe.ufrj.br (M.A.S. Neves).

Nomenclature

A_0	waterline area at average hull position
A_w	wave amplitude
Fn	Froude number
g	acceleration of gravity
$h_p(t)$	parametric excitation
I_{yy0}	second moment of inertia of waterline area
J_{xx}	transversal mass moment of inertia
J_{yy}	longitudinal mass moment of inertia
k	wave number
m	ship mass
U	ship speed of advance
x_{f0}	longitudinal co-ordinate of centroid of waterline
ρ	density of water
z, ϕ, θ	heave, roll and pitch non-linear motions
$\dot{z}, \dot{\phi}, \dot{\theta}$	heave, roll and pitch steady linear motions
ξ, φ, ϑ	heave, roll and pitch perturbations
λ	wave length
ζ	wave elevation
χ	wave incidence
ω_e	encounter frequency
ω_w	wave frequency
ω_{n4}	roll natural frequency

operational risks associated with head seas parametric resonance high lightening this potentially dangerous situation. In the case of fishing vessels, head sea parametric rolling has been discussed in Neves et al. [14]. A more recent example of parametric resonance in head seas has been reported on a PCTC ship [15].

There is a general understanding that for many ship designs the simulation models available are capable of reproducing with confidence the roll amplifications resulting from parametric resonance. But, unfortunately, there are some known cases in which the numerical models tend to over-predict the resonant rolling motions observed in experiments [16]. In these cases, the classical Mathieu type modelling, in which parametric excitation is assessed considering terms up to the second order, tends to give excessive excitation. The Authors attempts to reproduce the strong amplifications observed in the tests with a fishing vessel with a transom stern hull employing a second order non-linear mathematical model also indicated excessive amplifications in the simulations [17].

In an attempt to give a robust answer to this question and improve the quality of the simulations, the Authors have introduced a new third order mathematical model. The model describes the heave, roll and pitch motions of the vessel in a completely coupled way – up to third order, thus establishing a mathematically congruent set of equations.

In the present paper numerical simulations obtained from the derived third order non-linear mathematical model are compared to experimental results corresponding to excessive motions of a transom stern fishing vessel in head seas. The present investigation is limited to head seas configurations. It has been demonstrated recently that head seas parametric resonance may be a serious source of risk to ships [10].

It is shown that this new enhanced model gives a better comparison with the experiments than a second order model. Given the extended complexities of the coupled non-linear system, quite a lot of attention is devoted to interpreting the essential dynamic characteristics of the new mathematical model. Stability analysis carried out by means of the linear variational equation discloses the existence of super-harmonics and increased stiffness of the dynamic system – proportional to wave amplitude squared – due to third order coupling terms.

Limits of stability corresponding to the linear variational equation of the coupled roll motion are analytically derived. Additionally, numerical limits of stability corresponding to the non-linear equations are computed and compared to the analytically derived limits.

2. The non-linear heave – roll – pitch system of equations

Two right-handed co-ordinate systems are employed to describe the motions. An inertial reference frame (C, x, y, z) is assumed to be fixed at the mean ship motion, defined by the ship speed U . Regular waves are assumed to travel forming an angle χ with ship course. Another reference frame ($O, \bar{x}, \bar{y}, \bar{z}$) is fixed at the ship having the $\bar{x}\bar{y}$ plane coinciding, for the ship at rest, with the undisturbed sea surface, \bar{z} -axis passing through the vertical that contains the centre of gravity. The two systems coincide when excitations are absent (see Fig. 1).

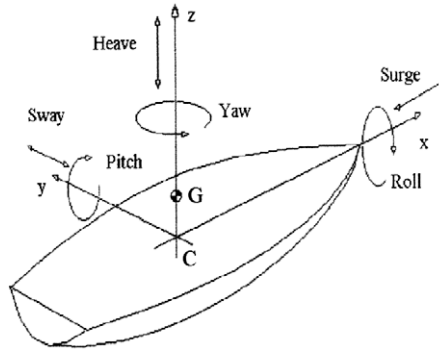


Fig. 1. Co-ordinate axis and definition of the six degrees of motions.

Non-linear equations of motion considering the three restoring degrees of freedom may be expressed in matrix form:

$$(\tilde{M} + \tilde{A})\ddot{\vec{s}} + \tilde{B}(\dot{\vec{s}})\dot{\vec{s}} + \tilde{C}_r(\vec{s}, \zeta) = \tilde{C}_{ext}(\zeta, \dot{\zeta}, \ddot{\zeta}), \quad (1)$$

where the displacement vector:

$$\vec{s}(t) = [z(t) \quad \phi(t) \quad \theta(t)]^T \quad (2)$$

defines the heave translational motion and the roll and pitch angular modes as indicated in Fig. 1.

In Eq. (1) \tilde{M} is a 3×3 diagonal matrix which describes hull inertia characteristics. Its elements are: m , the ship mass, J_{xx} , J_{yy} the mass moments of inertia in the roll and pitch modes, respectively, taken with reference to the chosen origin:

$$\tilde{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{yy} \end{bmatrix}.$$

\tilde{A} is also a 3×3 matrix, whose elements represent hydrodynamic generalized added masses. Following the reasoning of Abkowitz [18] and as usually adopted in seakeeping studies, these hydrodynamic reactions will be taken as linear. $\tilde{B}(\dot{\vec{s}})$ describes the coefficients of the hydrodynamic reactions dependent on ship velocities (damping). In conventional ship forms, damping in heave and pitch modes have demonstrated to be well represented by only linear terms, however, in roll, due to the great influence of viscosity, it may incorporate non-linear terms [19,20].

$\tilde{C}_r(\vec{s}, \zeta)$ is a 3×1 vector which describes non-linear restoring forces and moments dependent on the relative motions between ship hull and wave elevation $\zeta(t)$, as shown later in Eq. (5). On the right hand side of Eq. (1), the generalized vector $\tilde{C}_{ext}(\zeta, \dot{\zeta}, \ddot{\zeta})$ represents wave external excitation, usually referred to in the literature as the Froude-Krilov plus diffraction wave forcing terms, dependent on wave heading χ , encounter frequency ω_e , wave amplitude A_w and time t .

3. Hydrodynamic coefficients associated with potential theory

The hydrodynamic inertia and damping matrices are expressed, respectively, as:

$$\tilde{A} = \begin{bmatrix} Z_{\ddot{z}} & 0 & Z_{\ddot{\theta}} \\ 0 & K_{\dot{\phi}} & 0 \\ M_{\ddot{z}} & 0 & M_{\ddot{\theta}} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} Z_{\dot{z}} & 0 & Z_{\dot{\theta}} \\ 0 & K_{\phi}(\dot{\phi}) & 0 \\ M_{\dot{z}} & 0 & M_{\dot{\theta}} \end{bmatrix}. \quad (3)$$

It is generally accepted that, as mentioned in Section 2, with the exception of the roll damping coefficient, all the other terms in matrices \tilde{A} and \tilde{B} may be evaluated by means of linear potential theory [19,20]. The uncoupled nature of hydrodynamic reactions in the roll, on one side, and the heave and pitch modes on the other, is evident from the structure of the \tilde{A} and \tilde{B} matrices.

4. Non-linear roll damping

Roll damping moment cannot be satisfactorily computed by means of potential theory due to the occurrence of strong viscous effects. The semi-empirical model for prediction of roll damping moment proposed by Ikeda may be used to estimate the main linear and non-linear viscous effects [21]. The Ikeda model assumes that the total roll damping may be subdivided into five main components, each one being computed separately:

$$K_{\dot{\phi}}(\dot{\phi})\dot{\phi} = B_W + B_F + B_E + B_L + B_{BK},$$

where B_W , B_F , B_E , B_L and B_{BK} correspond to wave, friction, eddy, lift and bilge keel damping, respectively.

Bilge keel damping may also be split into components:

$$B_{BK} = B_{BKN} + B_{BKH} + B_{BKW},$$

where B_{BKN} , B_{BKH} and B_{BKW} are contributions due to normal force, interaction between hull and bilge keel and wave system generated by the bilge keels, respectively.

Mathematically, the roll damping moment may be represented as:

$$K_{\dot{\phi}}(\dot{\phi})\dot{\phi} = K_{\dot{\phi}}\dot{\phi} + K_{\dot{\phi}|\dot{\phi}}\dot{\phi}|\dot{\phi}|. \quad (4)$$

Linear and non-linear coefficients of Eq. (4) may be computed using expressions given in Himeno [21].

5. Non-linear restoring actions

Vector \vec{C}_r , representing non-linear restoring actions, will be expressed as:

$$\vec{C}_r = \begin{bmatrix} Z_r^{(1)} \\ K_r^{(1)} \\ M_r^{(1)} \end{bmatrix} + \begin{bmatrix} Z_{r(m)}^{(2)} + Z_{r(w)}^{(2)} \\ K_{r(m)}^{(2)} + K_{r(w)}^{(2)} \\ M_{r(m)}^{(2)} + M_{r(w)}^{(2)} \end{bmatrix} + \begin{bmatrix} Z_{r(m)}^{(3)} + Z_{r(w)}^{(3)} \\ K_{r(m)}^{(3)} + K_{r(w)}^{(3)} \\ M_{r(m)}^{(3)} + M_{r(w)}^{(3)} \end{bmatrix}, \quad (5)$$

where superscripts (1)–(3) refer to first, second and third order restoring terms, respectively. First order restoring actions correspond to the well-known hydrostatic actions:

$$\begin{aligned} Z_r^{(1)} &= Z_{zz} + Z_{\theta} = \rho g A_0 z - \rho g A_0 x_{f0} \theta, \\ K_r^{(1)} &= K_{\phi} = \rho g \nabla \overline{GM} \phi, \\ M_r^{(1)} &= M_{zz} + M_{\theta} = -\rho g A_0 x_{f0} z + \rho g \nabla \overline{GM_L} \theta. \end{aligned} \quad (6)$$

It is pointed out that the linear restoring action in the roll motion is governed by the so-called transversal metacentric height \overline{GM} . The roll natural frequency is defined as:

$$\omega_{n4} = \sqrt{\frac{K_{\phi}}{J_{xx} + K_{\dot{\phi}}}}.$$

Second and third order actions are composed of two terms, each. Subscripts (m) refer to body motions, whereas (w) refer to wave passage effects. Derivation of these actions, based on multivariable Taylor series expansions have been presented by Neves and Rodríguez [22,23]. According to the derivations, second order restoring actions (motions) may be defined as:

$$\begin{aligned} Z_{r(m)}^{(2)} &= \frac{1}{2} [Z_{zz} z^2 + 2Z_{z\theta} z \theta + Z_{\phi\phi} \phi^2 + Z_{\theta\theta} \theta^2], \\ K_{r(m)}^{(2)} &= K_{z\phi} z + K_{\phi\theta} \theta, \\ M_{r(m)}^{(2)} &= \frac{1}{2} [M_{zz} z^2 + 2M_{z\theta} z \theta + M_{\phi\phi} \phi^2 + M_{\theta\theta} \theta^2]. \end{aligned} \quad (7)$$

Analogously, for the second order restoring actions (wave passage),

$$\begin{aligned} Z_{r(w)}^{(2)} &= Z_{\zeta z}(t) z + Z_{\zeta \theta}(t) \theta, \\ K_{r(w)}^{(2)} &= K_{\zeta \phi}(t) \phi, \\ M_{r(w)}^{(2)} &= M_{\zeta z}(t) z + M_{\zeta \theta}(t) \theta, \end{aligned} \quad (8)$$

third order restoring actions (motions),

$$\begin{aligned} Z_{r(m)}^{(3)} &= \frac{1}{6} [Z_{zzz} z^3 + 3Z_{zz\theta} z^2 \theta + 3Z_{\theta\theta z} \theta^2 z + Z_{\theta\theta\theta} \theta^3 + 3Z_{\phi\phi z} z \phi^2 + 3Z_{\phi\phi\theta} \theta \phi^2], \\ K_{r(m)}^{(3)} &= \frac{1}{6} [K_{\phi\phi\phi} \phi^3 + 3K_{zz\phi} z^2 \phi + 3K_{\theta\theta\phi} \theta^2 \phi + 6K_{z\phi\theta} z \phi \theta], \\ M_{r(m)}^{(3)} &= \frac{1}{6} [M_{zzz} z^3 + 3M_{zz\theta} z^2 \theta + 3M_{\theta\theta z} \theta^2 z + M_{\theta\theta\theta} \theta^3 + 3M_{\phi\phi z} z \phi^2 + 3M_{\phi\phi\theta} \theta \phi^2] \end{aligned} \quad (9)$$

and, third order restoring actions (wave passage):

$$\begin{aligned} Z_{r(w)}^{(3)} &= Z_{\zeta\zeta z}(t) z + Z_{\zeta\zeta\theta}(t) z^2 + Z_{\zeta\zeta\theta}(t) \theta + Z_{\zeta z\theta}(t) z \theta + Z_{\phi\phi\zeta}(t) \phi^2 + Z_{\theta\theta\zeta}(t) \theta^2, \\ K_{r(w)}^{(3)} &= K_{\zeta\zeta\phi}(t) \phi + K_{\zeta z\phi}(t) z \phi + K_{\zeta\theta\phi}(t) \theta \phi, \\ M_{r(w)}^{(3)} &= M_{\zeta\zeta z}(t) z + M_{\zeta\zeta\theta}(t) z^2 + M_{\zeta\zeta\theta}(t) \theta + M_{\zeta z\theta}(t) z \theta + M_{\phi\phi\zeta}(t) \phi^2 + M_{\theta\theta\zeta}(t) \theta^2. \end{aligned} \quad (10)$$

Table 1a

Hydrostatic restoring coefficients (calm water) – second order

Heave	Roll	Pitch
$Z_{zz} = -2\rho g \int_L \bar{y} \frac{\partial y}{\partial z} dx$	$K_{zz} = 0$	$M_{zz} = 2\rho g \int_L \bar{x} \frac{\partial y}{\partial z} dx$
$Z_{z\phi} = 0$	$K_{z\phi} = -2\rho g \int_L \bar{y}^2 \frac{\partial y}{\partial z} dx$	$M_{z\phi} = 0$
$Z_{z\theta} = 2\rho g \int_L \bar{x} \frac{\partial y}{\partial z} dx$	$K_{z\theta} = 0$	$M_{z\theta} = -2\rho g \int_L \bar{x}^2 \frac{\partial y}{\partial z} dx$
$Z_{\phi\phi} = -2\rho g \int_L \bar{y}^2 \frac{\partial y}{\partial z} dx$	$K_{\phi\phi} = 0$	$M_{\phi\phi} = 2\rho g \int_L \bar{x} \bar{y}^2 \frac{\partial y}{\partial z} dx$
$Z_{\phi\theta} = 0$	$K_{\phi\theta} = 2\rho g \int_L \bar{x} \bar{y}^2 \frac{\partial y}{\partial z} dx$	$M_{\phi\theta} = 0$
$Z_{\theta\theta} = -2\rho g \int_L \bar{x}^2 \frac{\partial y}{\partial z} dx$	$K_{\theta\theta} = 0$	$M_{\theta\theta} = 2\rho g \int_L \bar{x}^3 \frac{\partial y}{\partial z} dx$

Table 1b

Hydrostatic restoring coefficients (calm water) – third order

Heave	Roll	Pitch
$Z_{zzz} = 0^a$	$Z_{zz\phi} = 0$	$Z_{zz\theta} = 0^a$
$Z_{\phi\phi z} = \rho g \left[4 \int_L \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + A_0 \right]$	$Z_{\phi\phi\phi} = 0$	$Z_{\phi\phi\theta} = -\rho g \left[4 \int_L \bar{x} \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + A_0 x_{f0} \right]$
$Z_{\theta\theta z} = 0^a$	$Z_{\theta\theta\phi} = 0$	$Z_{\theta\theta\theta} = 0^a$
Roll		
$K_{zzz} = 0$	$K_{zz\phi} = \rho g \left[4 \int_L \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + A_0 \right]$	$K_{zz\theta} = 0$
$K_{\phi\phi z} = 0$	$K_{\phi\phi\phi} = \rho g \left[8 \int_L \bar{y}^3 \left(\frac{\partial y}{\partial z} \right)^2 dx + 2I_{xx0} \right]$	$K_{\phi\phi\theta} = 0$
$K_{\theta\theta z} = 0$	$K_{\theta\theta\phi} = \rho g \left[4 \int_L \bar{x}^2 \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + I_{yy0} \right]$	$K_{\theta\theta\theta} = 0$
Pitch		
$M_{zzz} = 0^a$	$M_{zz\phi} = 0$	$M_{zz\theta} = 0^a$
$M_{\phi\phi z} = -\rho g \left[4 \int_L \bar{x} \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + A_0 x_{f0} \right]$	$M_{\phi\phi\phi} = 0$	$M_{\phi\phi\theta} = \rho g \left[4 \int_L \bar{x}^2 \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + I_{yy0} \right]$
$M_{\theta\theta z} = 0^a$	$M_{\theta\theta\phi} = 0$	$M_{\theta\theta\theta} = 0^a$
Heave-roll-pitch coupling		
$Z_{z\phi\theta} = 0$	$K_{z\phi\theta} = -\rho g \left[4 \int_L \bar{x} \bar{y} \left(\frac{\partial y}{\partial z} \right)^2 dx + A_0 x_{f0} \right]$	$M_{z\phi\theta} = 0$

^a These expressions were obtained analytically for the case of a ship with inclined wall side, corresponding to a good approximation in the case of ships of conventional forms, small displacements and smooth transversal curvatures ($\partial^2 \bar{y} / \partial z^2 \rightarrow 0$) at the considered waterline.

It is important to notice that in both cases, restoring actions due to motions and due to wave passage, depend on time. In the restoring actions due to motions this dependence is implicit in the displacements terms of heave, roll and pitch, i.e., $z(t)$, $\phi(t)$ and $\theta(t)$, while in the restoring actions due to wave passage, that dependence is also present in the wave passage restoring coefficients. Analytic expressions for the determination of all – linear and non-linear – coefficients (hydrostatic and wave effect) can be found in Neves and Rodríguez [22,23]. Linear hydrostatic coefficients are given in Eq. (6). Second and third order hydrostatic coefficients are given in Table 1. As shown in the restoring actions, there is a complete non-linear coupling between the three modes, an essential characteristic of the new mathematical model.

6. Wave passage effect

As stated in the previous section, restoring actions can be divided in restoring actions due to pure motions and restoring action to the wave passage. Physically, restoring actions due to motions represent the variations in the buoyant force and moments induced by the motions of the ships, and will exist even in the absence of surface waves. On the other hand, restoring actions due to wave passage represent the variations of buoyancy actions induced by the time-varying wave profile along the hull. As shown in Section 5, the wave passage effect is described by vectors $\vec{C}_{r(w)}^{(2)}$ and $\vec{C}_{r(w)}^{(3)}$ which include the incident wave elevation function described by $\zeta(t)$.

Incident wave elevation, according to the Airy linear theory, may be defined as [24]:

$$\zeta(x, y, t; \chi) = A_w \cos[kx \cos(\chi) + ky \sin(\chi) - \omega_e t], \quad (11)$$

where A_w is the wave amplitude; k , wave number; χ is wave heading (incidence); and ω_e is the encounter frequency, defined as: $\omega_e = \omega_w - kU \cos \chi$; ω_w is the wave frequency.

In longitudinal waves, head seas ($\chi = 180^\circ$), the equation of wave surface elevation is:

$$\zeta(x, t) = A_w \cos[kx + \omega_e t]. \quad (12)$$

It is clear that the wave coefficients are dependent on hull characteristics as well as on wave amplitude, frequency and time. These coefficients (in fact, periodic functions) may be expressed in terms of their cosine and sine terms, such that their dependence on wave amplitude becomes explicit. Thus, for example, in the case of roll motion, the second order term $K_{\zeta\phi}(t)$, proportional to the wave amplitude, may be expressed as:

Table 2

Coefficients of wave passage in the roll equation

$$K_{\zeta\phi c} = 2\rho g \int_L y^2 \frac{\partial y}{\partial z} \cos(kx) dx$$

$$K_{\zeta\phi s} = -2\rho g \int_L y^2 \frac{\partial y}{\partial z} \sin(kx) dx$$

$$K_{\zeta z\phi c} = -\rho g \int_L \left[4\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + 2\bar{y} \right] \cos(kx) dx$$

$$K_{\zeta z\phi s} = \rho g \int_L \left[4\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + 2\bar{y} \right] \sin(kx) dx$$

$$K_{\zeta\phi\theta c} = \rho g \int_L \left[4\bar{x}\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + 2\bar{x}\bar{y} \right] \cos(kx) dx$$

$$K_{\zeta\phi\theta s} = -\rho g \int_L \left[4\bar{x}\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + 2\bar{x}\bar{y} \right] \sin(kx) dx$$

$$K_{\zeta\zeta\phi c} = \rho g \int_L \left[\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + \frac{1}{2}\bar{y} \right] \cos(2kx) dx$$

$$K_{\zeta\zeta\phi s} = -\rho g \int_L \left[\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + \frac{1}{2}\bar{y} \right] \sin(2kx) dx$$

$$K_{\zeta\zeta\phi 0} = \rho g \int_L \left[\bar{y} \left(\frac{\partial y}{\partial z} \right)^2 + \frac{1}{2}\bar{y} \right] dx$$

$$K_{\zeta\phi}(t) = A_w [K_{\zeta\phi c} \cos(\omega_e t) + K_{\zeta\phi s} \sin(\omega_e t)]. \quad (13)$$

Physically, the associated term to this coefficient (13), represent the roll restoring moment acting on the ship when it is inclined an angle ϕ and a wave passes along it. The same reasoning applies to the other terms associated with the second order wave restoring coefficients.

The third order terms $K_{\zeta z\phi}(t)$ and $K_{\zeta\phi\theta}$, also proportional to wave amplitude, are expressed as:

$$K_{\zeta z\phi}(t) = A_w [K_{\zeta z\phi c} \cos(\omega_e t) + K_{\zeta z\phi s} \sin(\omega_e t)],$$

$$K_{\zeta\phi\theta}(t) = A_w [K_{\zeta\phi\theta c} \cos(\omega_e t) + K_{\zeta\phi\theta s} \sin(\omega_e t)]. \quad (14)$$

The first expression (14) is associated with the roll restoring moment acting on the ship when it is displaced in z meters in heave and inclined ϕ degrees in roll, and a wave passes along it. Analogously, the second expression (14) will be associated with the roll restoring moment acting on the ship when it is inclined simultaneously in roll and pitch and a wave passes along it. It must be observed that the role of these two functions is to parametrically excite the coupled system, appearing multiplied by $z(t)$ and $\theta(t)$, respectively, which on their turn are also periodic functions dependent on wave amplitude. The third order term $K_{\zeta\zeta\phi}(t)$, proportional to wave amplitude squared, is:

$$K_{\zeta\zeta\phi}(t) = A_w^2 [K_{\zeta\zeta\phi 0} + K_{\zeta\zeta\phi c} \cos(2\omega_e t) + K_{\zeta\zeta\phi s} \sin(2\omega_e t)], \quad (15)$$

where it should be observed that it is composed of a constant plus a super-harmonic with double the encounter frequency. Table 2 gives the cosine and sine parts of these coefficients, independent of wave amplitude and time.

Similar derivations have been presented by Paulling but with some simplifications in the third order terms [25]. The present derivation, Ref. [23], aggregates the various integral terms involving the longitudinal distribution of the product of half-breadth and the flare squared (see Table 2).

7. Non-linear equations of motion

Substitution of Eqs. (3)–(10) into Eq. (1) results in the following set of non-linear coupled equations of motion, with non-linearities defined to third order:

$$\begin{aligned} (m + Z_{\dot{z}})\ddot{z} + Z_{\dot{z}}\dot{z} + Z_{\ddot{\theta}}\ddot{\theta} + Z_{\dot{\theta}}\dot{\theta} + Z_z z + Z_{\theta}\theta + \frac{1}{2}Z_{zz}z^2 + \frac{1}{2}Z_{\phi\phi}\phi^2 + \frac{1}{2}Z_{\theta\theta}\theta^2 + Z_{z\theta}z\theta + \frac{1}{6}Z_{zzz}z^3 + \frac{1}{2}Z_{zz\theta}z^2\theta + \frac{1}{2}Z_{\phi\phi z}\phi^2z \\ + \frac{1}{2}Z_{\phi\phi\theta}\phi^2\theta + \frac{1}{2}Z_{\theta\theta z}\theta^2z + \frac{1}{6}Z_{\theta\theta\theta}\theta^3 + Z_{\zeta z}(t)z + Z_{\zeta\theta}(t)\theta + Z_{\zeta\zeta z}(t)z + Z_{\zeta\zeta\theta}(t)\theta + Z_{\zeta\zeta z}(t)z^2 + Z_{\zeta\zeta\theta}(t)\theta + Z_{\zeta\zeta z}(t)z\theta + Z_{\phi\phi\zeta}(t)\phi^2 \\ + Z_{\theta\theta\zeta}(t)\theta^2 = Z_w(t) \end{aligned} \quad (16)$$

$$\begin{aligned} (J_{xx} + K_{\ddot{\phi}})\ddot{\phi} + K_{\dot{\phi}}\dot{\phi} + K_{\phi|\dot{\phi}}|\dot{\phi}| + K_{\phi\dot{\phi}\dot{\phi}}\dot{\phi}^3 + K_{\phi}\phi + K_{z\phi}z\phi + K_{\phi\theta}\phi\theta + \frac{1}{2}K_{zz\phi}z^2\phi + \frac{1}{6}K_{\phi\phi\phi}\phi^3 + \frac{1}{2}K_{\theta\theta\phi}\theta^2\phi + K_{z\phi\theta}z\phi\theta \\ + K_{\zeta\phi}(t)\phi + K_{\zeta\zeta\phi}(t)\phi + K_{\zeta z\phi}(t)z\phi + K_{\zeta\phi\theta}(t)\phi\theta = K_w(t) \end{aligned} \quad (17)$$

$$\begin{aligned} (J_{yy} + M_{\ddot{\theta}})\ddot{\theta} + M_{\dot{\theta}}\dot{\theta} + M_z\ddot{z} + M_z\dot{z} + M_z z + M_{\theta}\theta + \frac{1}{2}M_{zz}z^2 + \frac{1}{2}M_{\phi\phi}\phi^2 + \frac{1}{2}M_{\theta\theta}\theta^2 + M_{z\theta}z\theta + \frac{1}{6}M_{zzz}z^3 + \frac{1}{2}M_{zz\theta}z^2\theta \\ + \frac{1}{2}M_{\phi\phi z}\phi^2z + \frac{1}{2}M_{\phi\phi\theta}\phi^2\theta + \frac{1}{2}M_{\theta\theta z}\theta^2z + \frac{1}{6}M_{\theta\theta\theta}\theta^3 + M_{\zeta z}(t)z + M_{\zeta\theta}(t)\theta + M_{\zeta\zeta z}(t)z + M_{\zeta\zeta\theta}(t)\theta + M_{\zeta\zeta z}(t)z^2 + M_{\zeta\zeta\theta}(t)\theta + M_{\zeta z\theta}(t)z\theta \\ + M_{\phi\phi\zeta}(t)\phi^2 + M_{\theta\theta\zeta}(t)\theta^2 = M_w(t) \end{aligned} \quad (18)$$

For regular waves of arbitrary heading the exciting force and moments will be of the following form:

$$Z_w(t) = Z_{w0} \cos(\omega_e t + \alpha_{w3}),$$

$$K_w(t) = K_{w0} \cos(\omega_e t + \alpha_{w4}),$$

$$M_w(t) = M_{w0} \cos(\omega_e t + \alpha_{w5}),$$

where Z_{w0} , K_{w0} and M_{w0} are the amplitudes of wave force in heave and moments in roll and pitch modes, respectively; and α_{w3} , α_{w4} and α_{w5} are the phase differences between the corresponding actions and the waves. These amplitudes are assumed, at each heading, to be linearly related to A_w , the wave amplitude.

8. Theoretical analysis of perturbed roll motions: linear variational equations

It is pointed out that in general parametric resonance is modelled in the literature considering uncoupled versions of the roll equation, with:

- vertical motions, $z(t)$ and $\theta(t)$, assumed to be purely harmonic, and
- roll parametric excitation defined with non-linearities up to the second order,
- different levels of non-linearities in the calm water $GZ(t)$ curve,

as discussed, for instance, in Refs. [3,8,9,16,17,26–28].

In the nomenclature adopted in the present paper, such a simplified model of parametric rolling would typically read as:

$$(J_{xx} + K_{\phi})\ddot{\phi} + K_{\phi}\dot{\phi} + K_{\phi|\phi}|\dot{\phi}| + K_{\phi}\phi + [K_{z\phi}z + K_{\phi\theta}\theta + K_{\zeta\phi}(t)]\phi + \frac{1}{6}K_{\phi\phi\phi}\phi^3 = 0, \quad (19)$$

where the three terms within brackets represent contributions to parametric excitation (to second order) due to the heave and pitch motions and to volumetric changes in the submerged hull produced by the wave passage, respectively. Clearly, all these contributions are proportional to the wave amplitude. Eq. (19) is sometimes referred as Mathieu–Duffing equation.

Stability of motions described by the non-linear set of equations (16)–(18) may be assessed by means of the variational system. In its linear form it may be derived under the assumption that the non-linear motions may be decomposed as the sum of steady oscillatory solutions plus some small perturbations:

$$\vec{s}(t) = \vec{s}_{ST}(t) + \vec{p}(t) = \begin{bmatrix} z(t) \\ \phi(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \hat{z}(t) \\ \hat{\phi}(t) \\ \hat{\theta}(t) \end{bmatrix} + \begin{bmatrix} \xi(t) \\ \varphi(t) \\ \vartheta(t) \end{bmatrix} = A_w \begin{bmatrix} \eta_3 \cos(\omega_e t + \alpha_z) \\ \eta_4 \cos(\omega_e t + \alpha_\phi) \\ \eta_5 \cos(\omega_e t + \alpha_\theta) \end{bmatrix} + \begin{bmatrix} \xi(t) \\ \varphi(t) \\ \vartheta(t) \end{bmatrix}, \quad (20)$$

where

$$\vec{s}_{ST}(t) = [\hat{z}(t) \quad \hat{\phi}(t) \quad \hat{\theta}(t)]^T,$$

represents the heave, roll and pitch well-known linear solutions, and η_3 , η_4 , η_5 , are the corresponding transfer functions. Perturbations in the heave, roll and pitch modes are defined as:

$$\vec{p}(t) = [\xi(t) \quad \varphi(t) \quad \vartheta(t)]^T.$$

The vector $(\vec{C}_r)_{var}$ corresponding to the linear variational part of $\vec{C}_r(t)$ is then derived as [29]:

$$(\vec{C}_r(t))_{var} = \sum \left. \frac{\partial \vec{C}_r(t)}{\partial \vec{s}} \right|_{\vec{s}=\vec{s}_{ST}} \vec{p}(t). \quad (21)$$

This vector represents the restoring actions in the linear variational equations, obtained in terms of a Taylor series expansion of the heave, roll and pitch perturbations in the vicinity of the steady linear responses (\vec{s}_{ST}). With this definition and making use of Eq. (20) into Eqs. (16)–(18), the equations of motion relative to the perturbed motions $\vec{p}(t) = [\xi(t) \quad \varphi(t) \quad \vartheta(t)]^T$ may be derived:

$$\begin{aligned} (m + Z_{\ddot{z}})\ddot{\xi} + Z_{\dot{z}}\dot{\xi} + Z_{\ddot{\theta}}\ddot{\vartheta} + Z_{\dot{\theta}}\dot{\vartheta} + Z_{\zeta}\xi + Z_{\theta}\vartheta + \left(Z_{zz}\hat{z} + Z_{z\theta}\hat{\theta} + \frac{1}{2}Z_{zzz}\hat{z}^2 + \frac{1}{2}Z_{zz\phi}\hat{\phi}^2 + \frac{1}{2}Z_{z\theta\theta}\hat{\theta}^2 + Z_{zz\theta}\hat{z}\hat{\theta} \right)\xi \\ + (Z_{\phi\phi}\hat{\phi} + Z_{z\phi}\hat{z}\hat{\phi} + Z_{\phi\phi\theta}\hat{\phi}\hat{\theta})\varphi + \left(Z_{z\theta}\hat{z} + Z_{\theta\theta}\hat{\theta} + \frac{1}{2}Z_{zz\theta}\hat{z}^2 + \frac{1}{2}Z_{z\phi\theta}\hat{\phi}^2 + \frac{1}{2}Z_{\theta\theta\theta}\hat{\theta}^2 + Z_{z\theta\theta}\hat{z}\hat{\theta} \right)\vartheta + [Z_{\zeta\zeta}(t) + Z_{\zeta\zeta\zeta}(t)]\xi \\ + 2Z_{\zeta z}(t)\hat{z} + Z_{\zeta z\theta}(t)\hat{\theta}\xi + Z_{\phi\zeta}(t)\varphi + [Z_{\theta\zeta}(t) + Z_{\zeta z\theta}(t) + Z_{\zeta z\theta}(t)\hat{z} + 2Z_{\theta\theta\zeta}(t)\hat{\theta}]\vartheta = 0 \end{aligned} \quad (22)$$

$$\begin{aligned} (J_x + K_{\phi})\ddot{\phi} + K_{\phi}\dot{\phi} + K_{\phi\phi}\hat{\phi}\dot{\phi} + K_{\phi}\phi + (K_{z\phi}\hat{z} + K_{z\phi\theta}\hat{\theta} + K_{z\phi\phi}\hat{\phi}\hat{\theta})\xi + \left(K_{z\phi}\hat{z} + K_{\phi\theta}\hat{\theta} + \frac{1}{2}K_{\phi\phi\phi}\hat{\phi} + \frac{1}{2}K_{zz\phi}\hat{z}^2 \right. \\ \left. + \frac{1}{2}K_{\phi\theta\theta}\hat{\theta}^2 + K_{z\phi\theta}\hat{z}\hat{\theta} \right)\varphi + (K_{\phi\phi}\hat{\phi} + K_{\phi\theta\theta}\hat{\theta} + K_{z\phi\theta}\hat{z}\hat{\theta})\vartheta + K_{\zeta\zeta}(t)\xi + [K_{\phi\zeta}(t) + K_{\zeta\zeta\phi}(t) + K_{\zeta z\phi}(t)\hat{z} + K_{\zeta\phi\theta}(t)\hat{\theta}]\varphi + K_{\theta\zeta}(t)\vartheta = 0 \end{aligned} \quad (23)$$

$$\begin{aligned}
& (J_y + M_{\theta})\ddot{\vartheta} + M_{\theta}\dot{\vartheta} + M_z\ddot{\xi} + M_z\dot{\xi} + M_z\dot{\xi} + M_{\theta}\vartheta + \left(M_{zz}\hat{z} + M_{z0}\hat{\theta} + \frac{1}{2}M_{zzz}\hat{z}^2 + \frac{1}{2}M_{z\phi\phi}\hat{\phi}^2 + \frac{1}{2}M_{z00}\hat{\theta}^2 + M_{zz0}\hat{z}\hat{\theta} \right)\xi \\
& + (M_{\phi\phi}\phi + M_{z\phi\phi}\hat{z}\hat{\phi} + M_{\phi\phi\theta}\hat{\phi}\hat{\theta})\varphi + \left(M_{z0}\hat{z} + M_{00}\hat{\theta} + \frac{1}{2}M_{zzz}\hat{z}^2 + \frac{1}{2}M_{\phi\phi\theta}\hat{\phi}^2 + \frac{1}{2}M_{000}\hat{\theta}^2 + M_{z00}\hat{z}\hat{\theta} \right)\vartheta + [M_{z\zeta}(t) + M_{\zeta\zeta}(t) \\
& + 2M_{\zeta z}(t)\hat{z} + M_{\zeta z0}(t)\hat{\theta}]\xi + M_{\phi\zeta}(t)\varphi + [M_{\theta\zeta}(t) + M_{\zeta\zeta\theta}(t) + M_{\zeta z\theta}(t)\hat{z} + 2M_{\theta\theta\zeta}(t)\hat{\theta}]\vartheta = 0.
\end{aligned} \quad (24)$$

Eqs. (22)–(24) form a set of time-dependent coupled equations, assumed to regulate, to first approximation, the stability of the dynamic system defined by the non-linear equations defined previously, Eqs. (16)–(18).

9. Parametric excitation of the roll variational equation in longitudinal waves

In the particular case of longitudinal waves the roll linear solution is zero. That is, $\hat{\phi} \equiv 0$. Hence, the linear variational equations in longitudinal waves are then derived as:

$$\begin{aligned}
& (m + Z_z)\ddot{\xi} + Z_z\dot{\xi} + Z_{\theta}\ddot{\vartheta} + Z_{\theta}\dot{\vartheta} + Z_z\dot{\xi} + Z_{\theta}\vartheta + \left(Z_{zz}\hat{z} + Z_{z0}\hat{\theta} + \frac{1}{2}Z_{zzz}\hat{z}^2 + \frac{1}{2}Z_{z00}\hat{\theta}^2 + Z_{zz0}\hat{z}\hat{\theta} \right)\xi + \left(Z_{z0}\hat{z} + Z_{00}\hat{\theta} + \frac{1}{2}Z_{zz0}\hat{z}^2 \right. \\
& \left. + \frac{1}{2}Z_{000}\hat{\theta}^2 + Z_{z00}\hat{z}\hat{\theta} \right)\vartheta + [Z_{z\zeta}(t) + Z_{\zeta\zeta}(t) + 2Z_{\zeta z}(t)\hat{z} + Z_{\zeta z0}(t)\hat{\theta}]\xi + [Z_{\theta\zeta}(t) + Z_{\zeta\zeta\theta}(t) + Z_{\zeta z\theta}(t)\hat{z} + 2Z_{\theta\theta\zeta}(t)\hat{\theta}]\vartheta = 0
\end{aligned} \quad (25)$$

$$(J_{xx} + K_{\phi})\ddot{\varphi} + K_{\phi}\dot{\varphi} + \left[K_{\phi} + K_{z\phi}\hat{z} + K_{\phi\theta}\hat{\theta} + \frac{1}{2}K_{zz\phi}\hat{z}^2 + \frac{1}{2}K_{00\phi}\hat{\theta}^2 + K_{z\phi\theta}\hat{z}\hat{\theta} + K_{\zeta\phi}(t) + K_{\zeta z\phi}(t)\hat{z} + K_{\zeta\phi\theta}(t)\hat{\theta} + K_{\zeta\zeta\phi}(t) \right]\varphi = 0 \quad (26)$$

$$\begin{aligned}
& (J_y + M_{\theta})\ddot{\vartheta} + M_{\theta}\dot{\vartheta} + M_z\ddot{\xi} + M_z\dot{\xi} + M_z\dot{\xi} + M_{\theta}\vartheta + \left(M_{zz}\hat{z} + M_{z0}\hat{\theta} + \frac{1}{2}M_{zzz}\hat{z}^2 + \frac{1}{2}M_{z00}\hat{\theta}^2 + M_{zz0}\hat{z}\hat{\theta} \right)\xi \\
& + \left(M_{z0}\hat{z} + M_{00}\hat{\theta} + \frac{1}{2}M_{zz0}\hat{z}^2 + \frac{1}{2}M_{000}\hat{\theta}^2 + M_{z00}\hat{z}\hat{\theta} \right)\vartheta + [M_{z\zeta}(t) + M_{\zeta\zeta}(t) + 2M_{\zeta z}(t)\hat{z} + M_{\zeta z0}(t)\hat{\theta}]\xi + [M_{\theta\zeta}(t) + M_{\zeta\zeta\theta}(t) \\
& + M_{\zeta z\theta}(t)\hat{z} + 2M_{\theta\theta\zeta}(t)\hat{\theta}]\vartheta = 0.
\end{aligned} \quad (27)$$

It should be observed that in the case of longitudinal waves, Eqs. (25)–(27), the perturbed heave and pitch motions decouple from the roll mode, which becomes dependent on the heave and pitch steady oscillatory solutions. These heave and pitch linear responses are defined as:

$$\begin{aligned}
\hat{z}(t) &= A_w\eta_3 \cos(\omega_e t + \alpha_z), \\
\hat{\theta}(t) &= A_w\eta_5 \cos(\omega_e t + \alpha_{\theta}).
\end{aligned} \quad (28)$$

Substituting expressions (13)–(15) and (28) into Eq. (26) and conveniently rearranging terms results in the following expression for the roll linear variational equation in longitudinal waves:

$$(J_{xx} + K_{\phi})\ddot{\varphi} + K_{\phi}\dot{\varphi} + \left[K_{\phi} + A_w^2 K_0 + A_w(K_{1C} \cos \omega_e t + K_{1S} \sin \omega_e t) + A_w^2(K_{2C} \cos 2\omega_e t + K_{2S} \sin 2\omega_e t) \right]\varphi = 0, \quad (29)$$

where

$$\begin{aligned}
K_0 &= \frac{1}{4}K_{zz\phi}\eta_3^2 + \frac{1}{4}K_{\theta\theta\phi}\eta_5^2 + \frac{1}{2}K_{z\phi\theta}\eta_3\eta_5 \cos(\alpha_z - \alpha_{\theta}) + \frac{\eta_3}{2}[K_{\zeta z\phi C} \cos(\alpha_z) - K_{\zeta z\phi S} \sin(\alpha_z)] \\
&+ \frac{\eta_5}{2}[K_{\zeta\theta\phi C} \cos(\alpha_{\theta}) - K_{\zeta\theta\phi S} \sin(\alpha_{\theta})] + K_{\zeta\zeta\phi 0},
\end{aligned} \quad (30)$$

$$K_{1C} = -K_{z\phi}\eta_3 \cos(\alpha_z) + K_{\phi\theta}\eta_5 \cos(\alpha_{\theta}) + K_{\zeta\phi C}, \quad (31)$$

$$K_{1S} = -K_{z\phi}\eta_3 \sin(\alpha_z) - K_{\phi\theta}\eta_5 \sin(\alpha_{\theta}) + K_{\zeta\phi S}, \quad (32)$$

$$\begin{aligned}
K_{2C} &= \frac{1}{4}K_{zz\phi}\eta_3^2 \cos(2\alpha_z) + \frac{1}{4}K_{\theta\theta\phi}\eta_5^2 \cos(2\alpha_{\theta}) + \frac{1}{2}K_{z\phi\theta}\eta_3\eta_5 \cos(\alpha_z + \alpha_{\theta}) \\
&+ \frac{\eta_3}{2}[K_{\zeta z\phi C} \cos(\alpha_z) + K_{\zeta z\phi S} \sin(\alpha_z)] + \frac{\eta_5}{2}[K_{\zeta\theta\phi C} \cos(\alpha_{\theta}) + K_{\zeta\theta\phi S} \sin(\alpha_{\theta})] + K_{\zeta\zeta\phi C},
\end{aligned} \quad (33)$$

$$\begin{aligned}
K_{2S} &= -\frac{1}{4}K_{zz\phi}\eta_3^2 \sin(2\alpha_z) - \frac{1}{4}K_{\theta\theta\phi}\eta_5^2 \sin(2\alpha_{\theta}) - \frac{1}{2}K_{z\phi\theta}\eta_3\eta_5 \sin(\alpha_z + \alpha_{\theta}) \\
&+ \frac{\eta_3}{2}[-K_{\zeta z\phi C} \sin(\alpha_z) + K_{\zeta z\phi S} \cos(\alpha_z)] + \frac{\eta_5}{2}[-K_{\zeta\theta\phi C} \sin(\alpha_{\theta}) + K_{\zeta\theta\phi S} \cos(\alpha_{\theta})] + K_{\zeta\zeta\phi S},
\end{aligned} \quad (34)$$

in which it is possible to identify the distinct third order contributions to parametric excitation, from purely hydrostatic actions and wave interaction effects, respectively. It is observed in Eq. (29) that its time-dependent part is the sum of a simple harmonic and a bi-harmonic term. Therefore, it does not correspond to a Mathieu type equation. The Mathieu equation displays only simple harmonic time-dependent restoring effects and describes the type of stability associated with the

variational equation of non-linear systems of second order. Restoring terms of the third order give rise not only to the bi-harmonic terms, but also to the non-periodic term that accompanies the natural frequency. The consequence is that now the linear variational equations correspond to a set of coupled Hill equations.

10. Limits of stability – analytical approach

The stability of solutions corresponding to coupled Hill-type equations has been investigated by Hsu [30]. In the case of Eq. (29), two regions of instability appear. In accordance with the nomenclature adopted in the present work, the expressions for the curves of the limits of these instability regions are [30]:

- First region of stability ($s = 1$):

$$2\omega_4 + \frac{A_w}{2\omega_4} \left[\left(d_{44}^{(1)} \right)^2 + \left(e_{44}^{(1)} \right)^2 - 4\omega_4^2 \left(f_{44}^{(0)} \right)^2 \right]^{\frac{1}{2}} > \omega_e > 2\omega_4 - \frac{A_w}{2\omega_4} \left[\left(d_{44}^{(1)} \right)^2 + \left(e_{44}^{(1)} \right)^2 - 4\omega_4^2 \left(f_{44}^{(0)} \right)^2 \right]^{\frac{1}{2}}. \quad (35)$$

- Second region of stability ($s = 2$):

$$\omega_4 + \frac{A_w}{4\omega_4} \left[\left(d_{44}^{(1)} \right)^2 + \left(e_{44}^{(1)} \right)^2 - 4\omega_4^2 \left(f_{44}^{(0)} \right)^2 \right]^{\frac{1}{2}} > \omega_e > \omega_4 - \frac{A_w}{4\omega_4} \left[\left(d_{44}^{(1)} \right)^2 + \left(e_{44}^{(1)} \right)^2 - 4\omega_4^2 \left(f_{44}^{(0)} \right)^2 \right]^{\frac{1}{2}}, \quad (36)$$

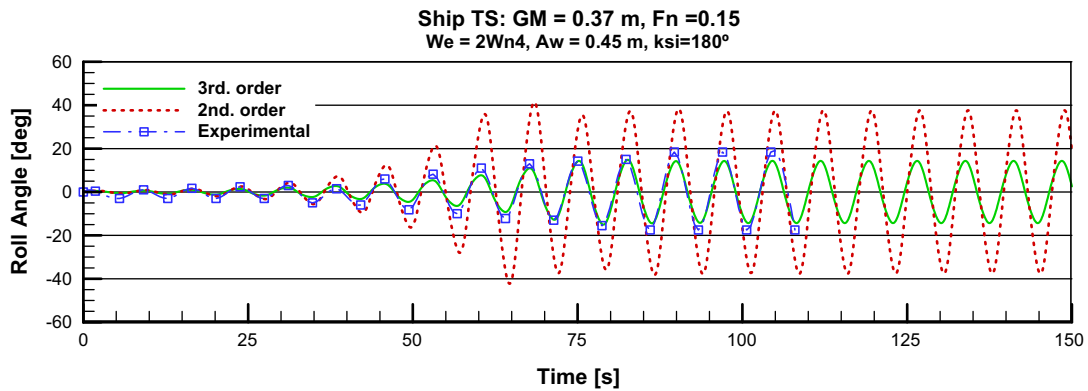
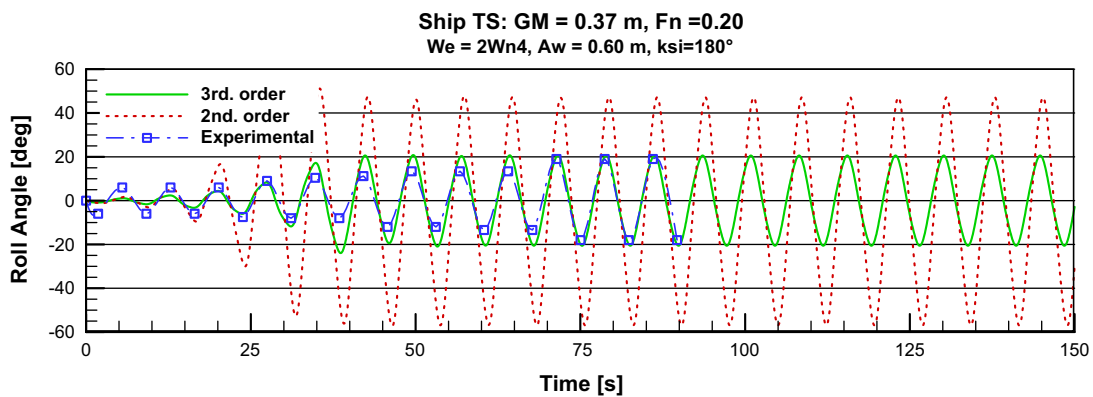
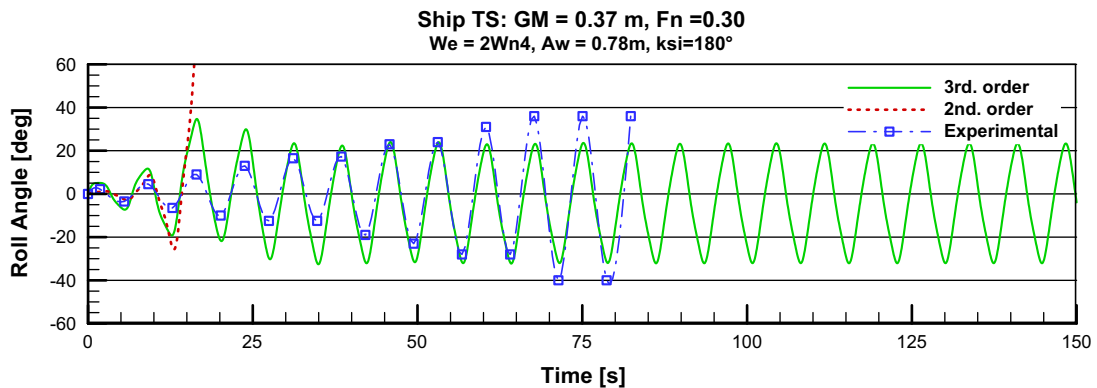
where

$$\begin{aligned} d_{44}^{(1)} &= \frac{[K_{z\phi}\eta_3 \cos(\alpha_z) + K_{\phi\theta}\eta_5 \cos(\alpha_\theta) + K_{\zeta\phi c}]}{(J_{xx} + K_{\dot{\phi}})}, \\ e_{44}^{(1)} &= \frac{[-K_{z\phi}\eta_3 \sin(\alpha_z) - K_{\phi\theta}\eta_5 \sin(\alpha_\theta) + K_{\zeta\phi s}]}{(J_{xx} + K_{\dot{\phi}})}, \\ d_{44}^{(2)} &= \frac{A_w}{(J_{xx} + K_{\dot{\phi}})} \left[\frac{1}{4} K_{zz\phi}\eta_3^2 \cos(2\alpha_z) + \frac{1}{4} K_{\theta\theta\phi}\eta_5^2 \cos(2\alpha_\theta) + \frac{1}{2} K_{z\phi\theta}\eta_3\eta_5 \cos(\alpha_z + \alpha_\theta) \right. \\ &\quad \left. + \frac{\eta_3}{2} [K_{\zeta z\phi c} \cos(\alpha_z) + K_{\zeta\phi s} \sin(\alpha_z)] + \frac{\eta_5}{2} [K_{\zeta\theta\phi c} \cos(\alpha_\theta) + K_{\zeta\theta\phi s} \sin(\alpha_\theta)] + K_{\zeta\zeta\phi c} \right], \\ e_{44}^{(2)} &= \frac{A_w}{(J_{xx} + K_{\dot{\phi}})} \left[-\frac{1}{4} K_{zz\phi}\eta_3^2 \sin(2\alpha_z) - \frac{1}{4} K_{\theta\theta\phi}\eta_5^2 \sin(2\alpha_\theta) - \frac{1}{2} K_{z\phi\theta}\eta_3\eta_5 \sin(\alpha_z + \alpha_\theta) \right. \\ &\quad \left. + \frac{\eta_3}{2} [-K_{\zeta z\phi c} \sin(\alpha_z) + K_{\zeta z\phi s} \cos(\alpha_z)] + \frac{\eta_5}{2} [-K_{\zeta\theta\phi c} \sin(\alpha_\theta) + K_{\zeta\theta\phi s} \cos(\alpha_\theta)] + K_{\zeta\zeta\phi s} \right], \\ f_{44}^{(0)} &= \frac{K_{\dot{\phi}}}{A_w(J_{xx} + K_{\dot{\phi}})}, \\ \omega_4 &= [\omega_{n4}^2 + \omega_{m4}^2]^{1/2}; \quad \omega_{n4}^2 = \frac{K_{\dot{\phi}}}{(J_{xx} + K_{\dot{\phi}})}, \\ \omega_{m4}^2 &= \frac{A_w^2}{(J_{xx} + K_{\dot{\phi}})} \left[\frac{1}{4} K_{zz\phi}\eta_3^2 + \frac{1}{4} K_{\theta\theta\phi}\eta_5^2 + \frac{1}{2} K_{z\phi\theta}\eta_3\eta_5 \cos(\alpha_z - \alpha_\theta) \right. \\ &\quad \left. + \frac{\eta_3}{2} [K_{\zeta z\phi c} \cos(\alpha_z) - K_{\zeta z\phi s} \sin(\alpha_z)] + \frac{\eta_5}{2} [K_{\zeta\theta\phi c} \cos(\alpha_\theta) - K_{\zeta\theta\phi s} \sin(\alpha_\theta)] + K_{\zeta\zeta\phi 0} \right]. \end{aligned}$$

11. Numerical simulations – results and discussions

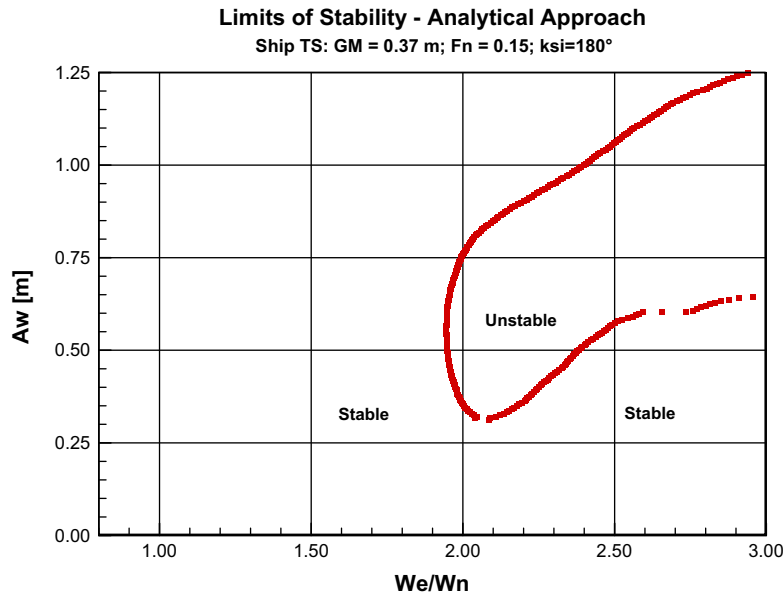
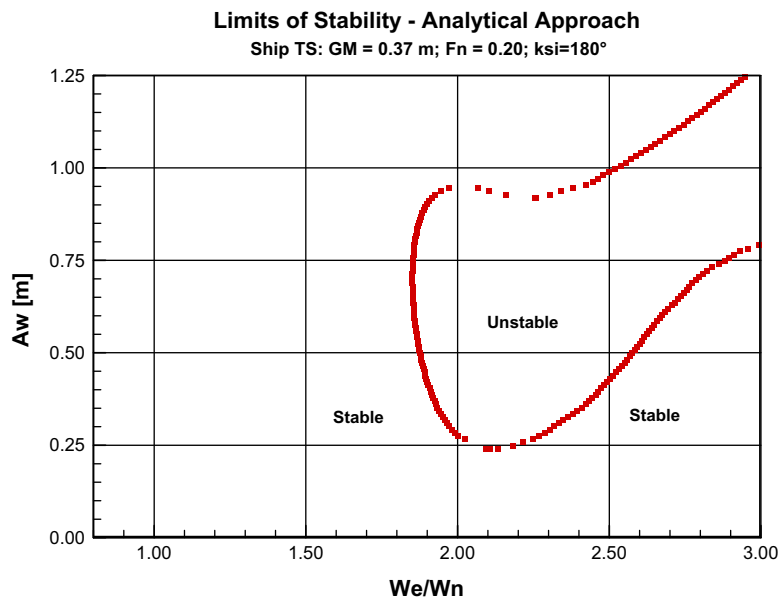
After computing all the derivative coefficients of Eqs. (16)–(18) for a given hull form in accordance with the expressions given in Tables 1 and 2, numerical simulations of the non-linear equations of motions may be performed. Results are given below for a transom stern (TS) fishing vessel, which has been tested in head seas [14]. Main characteristics of the TS hull are described in the Appendix. Figs. 2–4 show roll responses for ship TS integrated in time domain applying a 4th order Runge–Kutta algorithm.

The numerical simulations of the ship roll responses presented in Figs. 2–4 correspond to a loading condition of $\overline{GM} = 0.37$ m, see Ref. [14], at the exact Mathieu tuning in the first region of instability ($\omega_e = 2\omega_{n4}$) for different speeds of advance and wave amplitudes. The corresponding experimental time series are also included in the figures, and are compared with the numerical results of the present model. Additionally, in order to assess the effects of the 3rd order terms, the responses corresponding to the classical 2nd order model (based on the Mathieu equation) are also given. It is observed that in general the third order model gives much better agreement with the experiments than the Mathieu-based model (second

Fig. 2. Roll response, $F_n = 0.15$, $A_w = 0.45$ m.Fig. 3. Roll response, $F_n = 0.20$, $A_w = 0.60$ m.Fig. 4. Roll response, $F_n = 0.30$, $A_w = 0.78$ m.

order model). In general, the second order model tends to over-predict parametric rolling and in some cases fails to reproduce it, showing unbounded responses (Fig. 4). In general, the third order model gives a quite satisfactory agreement with the experiments. The better agreement of the third order model in these cases is explained by the existence of the non-linear rigidity which makes the system less prone to parametric excitation.

Figs. 5 and 6 show the results for the analytical limits of stability based on the numerical implementation of expressions (35) and (36) for the case of a transom stern fishing vessel for two forward speeds. These results are based on the roll variational equation and allow some qualitative analysis with interesting results and conclusions about the dynamic characteristics of the roll motion described by third order mathematical model:

Fig. 5. Analytical approach, $Fn = 0.15$.Fig. 6. Analytical approach, $Fn = 0.20$.

- First zone of instability ($\omega_e/\omega_{n4} = 2$) is wider, therefore, potentially more dangerous than the subsequent one, as had already been observed by Blocki [3], Skomedal [26], and others in the analysis of Mathieu type equations.
- In a second order model, rigidity is directly proportional to the GM value. In the third order model, there is an additional contribution of non-linear origin (K_0) having a quadratic dependence on the wave amplitude. A more rigid system means less percentage of change of the instantaneous restoring moment with respect to the calm water restoring moment (initial rigidity), thus reducing the sensitivity of the dynamic system to parametric resonance. This tendency had been observed by Skomedal [26] in pure numerical analyses.
- Bending to the right of the stability curves in the third order model, not observed in the classical second order models (Mathieu-based). This is due to the non-linear stiffening of the system. As expected, it is more significant for higher waves.
- *Exact* tuning in the first region of instability, which in the second order model is defined by $\omega_e/\omega_{n4} = 2$, is represented by a vertical backbone line, whereas in the third order model this is given by a backbone curve defined by $\omega_e/\omega_4 = 2$, where ω_4

is the non-linear frequency of oscillation of the system in waves. For waves with higher amplitudes, the curve of *exact* tuning bends to the right introducing a detuning to the system, as a function of wave amplitude squared.

- It should be observed that in some cases, as in Figs. 5 and 6, the areas of instability display upper frontiers, as a consequence of the bending to the right of the stability curves. That implies that at a given frequency, above a certain level of wave amplitude, the increase in rigidity defuses the increase in the amplitude of parametric excitation.

An alternative way of computing the instability regions is by solving the non-linear equations of motion, Eqs. (16)–(18), for a large set of wave amplitudes and tuning factors (encounter frequency/natural roll frequency), which may be varied systematically. Then, at each time that instabilization takes place (roll amplification), a point may be plotted in the corresponding plane (A_w vs. tuning factor). Depending on the magnitude of the steady roll amplitude, these points have an identifying

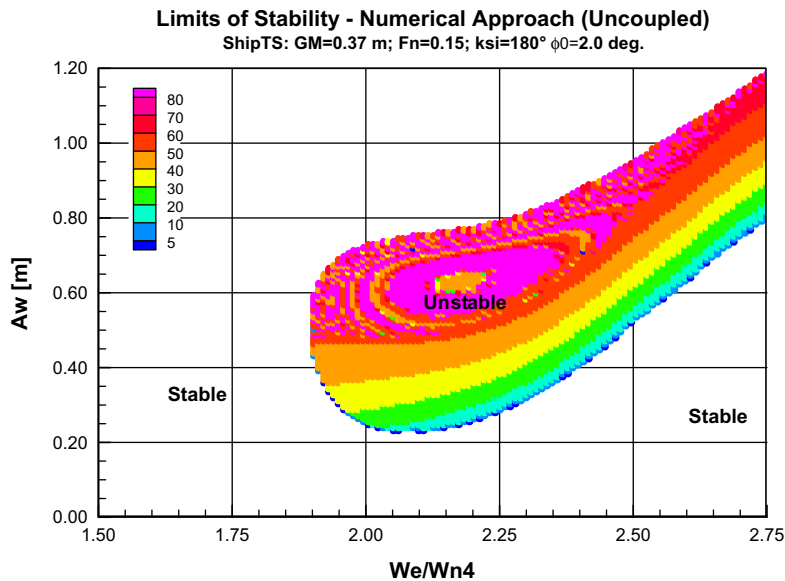


Fig. 7. Numerical uncoupled approach, $F_n = 0.15$.

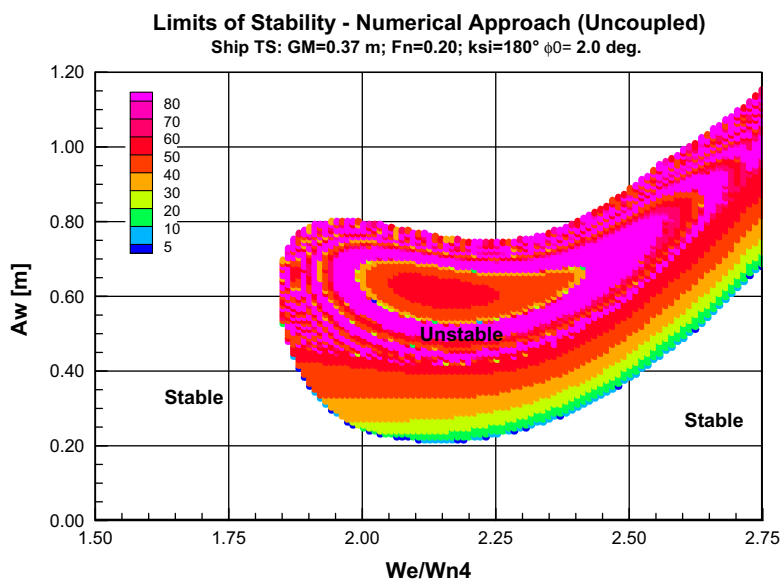


Fig. 8. Numerical uncoupled approach $F_n = 0.20$.

color. In spite of the aspect that this procedure is much more time consuming for computation, it has the advantage of letting us know the instability regions not only qualitative, but also quantitatively. Under this alternative method, two different numerical approaches could be used: one assuming an uncoupled non-linear roll motion equation, and the other using the 3 DOF motion equations coupling the heave, roll and pitch modes.

11.1. Numerical approach – uncoupled roll motion

Suppose that parametric rolling is modelled as an uncoupled mode from the vertical motions. Numerical results may be obtained for the roll motion amplitude assuming that the heave and pitch motions are linear harmonic functions in Eq. (17). It is observed that the numerical results are quite similar to the analytical values in terms of the area of instabilities.

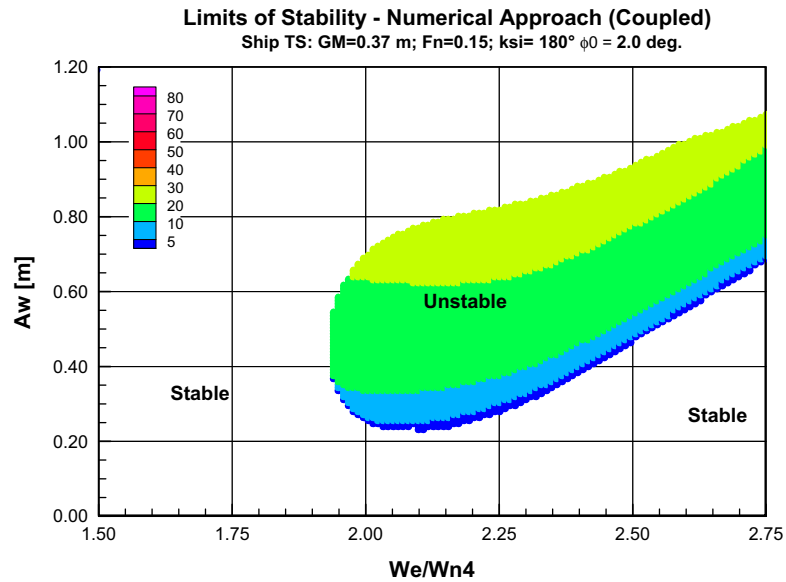


Fig. 9. Numerical 3 DOF approach, $F_n = 0.15$ ($\phi_0 = 2^\circ$).

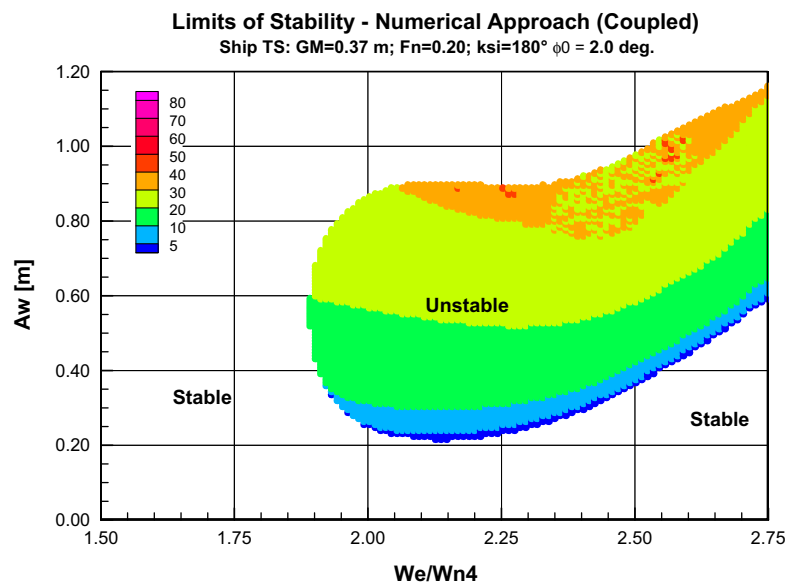


Fig. 10. Numerical 3 DOF approach, $F_n = 0.20$ ($\phi_0 = 2^\circ$).

As can be noted from Figs. 7 and 8, the shape and location of the regions of stability agree well with the analytical approach. This would indicate that pure roll non-linearities have little or NO influence on these characteristics of the limits. On the other hand, roll results inside the unstable area are relevant information made available in this numerical procedure.

11.2. Numerical approach – coupled 3 DOF motions

A more refined and reliable way of getting the limits of stability for parametric resonance is to solve numerically the three-degrees-of freedom (DOF) ship motion Eq. (2), and plot the responses, as explained previously. This more complete approach when applied to the same conditions tested above for TS ship resulted in the limits of stability shown in Figs. 9 and 10. In general, the shape and location of first instability regions ($\omega_e = 2.0\omega_{n4}$) obtained with the 3 DOF numerical approach agrees well with the previous approaches. However, when comparing the limits of Figs. 9 and 10 with the corresponding ones of Figs. 7 and 8, relevant differences can be observed in the mapping of amplitudes of parametric rolling for the two forward speeds. Such differences reflect the influence of non-linearities of heave and pitch, which in general tend to control the magnitude of roll amplifications.

12. Conclusions

A derivative mathematical model was introduced, in which the heave, roll and pitch motions and wave passage effects were described with coupling terms up to the third order.

The 3rd order numerical simulations based on the proposed model show quite realistic comparisons with the experimental time series results in conditions for which the second order model offers poor results. This has been verified for distinct wave amplitudes and ship speeds.

In order to gain a deeper understanding of the complex non-linear responses to specific hull design configurations, a qualitative analysis of the coupled non-linear system has been performed. The analysis points out to the appearance of super-harmonics and a constant term due to heave and pitch motions coupling with roll and wave passage. Analytic expressions have been derived for these contributions. The additional terms have been computed and found to be positive contributions. It is thus concluded that the essential dynamic characteristics of the coupled system are those of a Hill equation with a hardening term proportional to wave amplitude squared.

The increased stiffness, proportional to wave amplitude squared, is thought to be responsible for giving to the third order model such a realistic description of the parametric amplification.

Considering that limits of stability are a practical and direct way of assessing the safety of a design, the paper introduced analytical expressions for the limits of stability of the roll linear variational equation. An interesting aspect of the new limits of stability may display upper frontiers, as a consequence of the bending to the right of the stability curves. That implies that at a given frequency, above a certain level of wave amplitude, further increasing the wave amplitude may imply in a complete disappearance of parametric amplification.

The limits of stability derived from the analytical approach have shown in general a good agreement with the corresponding limits obtained employing numerical procedures, and due to its relatively easy implementation, should find good applicability in the ship preliminary design stage. One limitation of the analytical approach is that it does not provide information on the magnitude and distribution of parametric rolling. To overcome this inconvenience, two numerical approaches have been proposed. One using the uncoupled roll equation, and the other applying the full non-linear equations coupling heave, roll and pitch. Comparing the two latter approaches, a relevant conclusion can be drawn, i.e., the extreme importance of non-linear couplings between the vertical modes and roll in the determination of parametric roll amplitudes. As can be noted in the respective figures, the uncoupled numerical approach can produce misleading predictions of parametric amplitudes.

Finally, it may be concluded that the proposed extension of the mathematical model to a coupled third order non-linear system introduces new and relevant dynamical characteristics into the resulting motions.

Acknowledgement

The present investigation is supported by CNPq within the STAB project (Non-linear Stability of Ships). The authors also acknowledge financial support from CAPES and FAPERJ.

Appendix. Main characteristics of the tested hulls

The main characteristics and the lines plans of the vessel used in the present research are shown in Table A.1 and Fig. A1, respectively. The hull corresponds to a typical transom stern hull, here denominated TS.

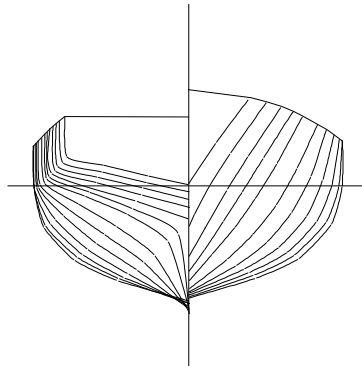
As mentioned above, these vessels have been tested experimentally under parametric rolling in longitudinal regular head sea waves at the first region of resonance, defined as the condition corresponding to the encounter frequency coinciding with twice the roll natural frequency. Detailed descriptions of all tested conditions can be found in Neves et al. [14].

Fig. A2 shows the static stability curve for the TS ship in a low GM condition. The figure also shows the corresponding polynomial fittings used in the numerical simulations.

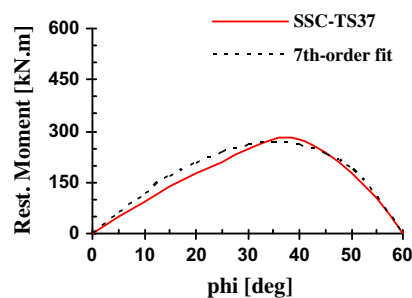
Table A.1

Principal particulars of ship

Denomination	TS
Length overall (m)	25.91
Length between perpend (m)	22.09
Beam (m)	6.86
Depth (m)	3.35
Draught (m)	2.48
Displacement (tons)	170.30
Water plane area (m ²)	121.00
Pitch radius of gyration (m)	5.35

**Fig. A1.** Lines plan of TS bull.

Static Stability Curve - ShipTS (GM = 0.37 m)

**Fig. A2.** Restoring curves for TS ship, GM = 0.37 m.

References

- [1] J.E. Kerwin, Notes on rolling in longitudinal waves, *Int. Shipbuild. Progress* 2 (16) (1955) 597–614.
- [2] J.R. Paulling, R.M. Rosenberg, On unstable ship motions resulting from non-linear coupling, *J. Ship Res.* 3 (1) (1959) 36–46.
- [3] W. Blocki, Ship safety in connection with parametric resonance of the roll, *Int. Shipbuild. Progress* 27 (306) (1980) 36–53.
- [4] J.O. De Kat, J.R. Paulling, The simulation of ship motions and capsizing in severe seas, *Trans. Soc. Naval Archit. Marine Engrs.* 97 (1989).
- [5] A. Munif, N. Umeda, Modeling extreme roll motions and capsizing of a moderate-speed ship in astern waves, *J. Soc. Naval Archit. Jpn.* 187 (2000).
- [6] O.H. Oakley, J.R. Paulling, P.D. Wood, Ship motions and capsizing in astern seas, in: *Proceedings of 10th Symposium on Naval Hydrodynamics*, Cambridge, MA, 1974.
- [7] N. Umeda, M. Hamamoto, et al., Model experiments of ship capsize in astern seas, *J. Soc. Naval Archit. Jpn.* 177 (1995) 207–217.
- [8] K.J. Spyrou, Designing against parametric instability in following seas, *Ocean Eng.* 27 (6) (2000) 625–653.
- [9] L. Arnold, I. Chueshov, G. Ochs, Stability and capsizing of ships in random sea – a survey. Tech. Report 464, Universitat Bremen Institut für Dynamicsche Systeme, 2003.
- [10] W.N. France, M. Levadou, T.W. Treake, J.R. Paulling, R.K. Michel, C. Moore, An investigation of head-sea parametric rolling and its influence on container lashing systems, *Marine Technol.* 40 (1) (2003) 1–19.
- [11] R.P. Dallinga, J.J. Blok, H.R. Luth, Excessive rolling of cruise ships in head and following waves, in: *RINA International Conference on Ship Motions and Manoeuvrability*, Royal Institute of Naval Architects, London, 1998.
- [12] H.R. Luth, R.P. Dallinga, Prediction of excessive rolling of cruise vessels in head and following waves, in: *Proceedings of PRADS Conference*, 1999.

- [13] M. Levadou, L. Palazzi, Assessment of operational risks of parametric roll, in: *Proceedings of World Maritime Technology Conference (WMTC'2003)*, Society of Naval Architects and Marine Engineers, San Francisco, 2003.
- [14] M.A.S. Neves, N. Pérez, O. Lorca, Experimental analysis on parametric resonance for two fishing vessels in head seas, in: *Proceedings of Sixth International Ship Stability Workshop*, Webb Institute, New York, 2002.
- [15] M. Palmquist, C. Nygren, Recordings of head-sea parametric rolling on a PCTC. Annex of Report of Document SLF 47/6/6-IMO, 2004.
- [16] N. Umeda, H. Hashimoto, D. Vassalos, S. Urano, K. Okou, Non-linear dynamics on parametric roll resonance with realistic numerical modelling, in: *Proceedings of Eighth International Conference on the Stability of Ships and Ocean Vehicles (STAB'2003)*, Madrid, 2003, pp. 281–290.
- [17] M.A.S. Neves, N.A. Pérez, L. Valerio, Stability of small fishing vessels in longitudinal waves, *Ocean Eng.* 26 (12) (1999) 1389–1419.
- [18] M.A. Abkowitz, *Stability and Motion Control of Ocean Vehicles*, The MIT Press, 1970.
- [19] N. Salvesen, O.E. Tuck, O. Faltinsen, Ship motions and sea loads, *Trans. SNAME* 78 (1970) 250–287.
- [20] W.G. Meyers, D.J. Sheridan, N. Salvesen, *Manual – NSRDC ship-motion and sea-load computer program*. Report No. 3376. Naval Ship Research and Development Centre, Maryland, 1975.
- [21] Y. Himeno, Prediction of ship roll damping – state of the art. Report No. 239, Department of Naval Architecture and Marine Engineering, The University of Michigan, 1981.
- [22] M.A.S. Neves, C. Rodríguez, A non-linear mathematical model of higher order for strong parametric resonance of the roll motion of ships in waves. *Marine Systems and Ocean Technology*, J. Sociedade Brasileira de Engenharia Naval 1 (2) (2005).
- [23] M.A.S. Neves, C. Rodríguez, On unstable ship motions resulting from strong non-linear coupling, *Ocean Eng.* 33 (14–15) (2006) 1853–1883.
- [24] J.N. Newman, *Marine Hydrodynamics*, The MIT Press, 1977.
- [25] J.R. Paulling, The transverse stability of a ship in a longitudinal seaway, *J. Ship Res.* 4 (4) (1961) 37–49.
- [26] N. Skomedal, Parametric excitation of roll motion and its influence on stability, in: *Proceedings of the Second International Conference on Stability of Ships and Ocean Vehicles (STAB'82)*, Tokyo, 1982, pp. 113–125.
- [27] J. Matuziak, On the effects of wave amplitude, damping and initial conditions on the parametric roll resonance, in: *Proceedings of Eighth International Conference on the Stability of Ships and Ocean Vehicles (STAB'2003)*, Madrid, 2003, pp. 341–347.
- [28] G. Bulian, A. Francescutto, C. Lugni, On the non-linear modelling of parametric rolling in regular and irregular waves, in: *Proceedings of Eighth International Conference on the Stability of Ships and Ocean Vehicles (STAB'2003)*, Madrid, 2003, pp. 305–323.
- [29] L. Cesari, *Asymptotic behaviour and stability problems in ordinary differential equations*, third ed., Springer-Verlag Berlin Heidelberg, Berlin, 1971.
- [30] C.S. Hsu, On the parametric excitation of a dynamic system having multiple degrees of freedom, *J. Appl. Mech.* 30 (3) (1963) 367–372.