Phys 300 - Homework II

Assigned: March 13, 2012, Tuesday.
Due: March 21, 2012,Wednesday, at 3:30 pm.

1. Let $\psi(x)$ denote the position-space wavefunction and $g(k)$ is the momentum space wavefunction. Remember that they are related as follows

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{ikx} dk \quad . \tag{1}$$

Show that

$$\int \psi^*(x) x \psi(x) dx = \int g^*(k) i \frac{\partial}{\partial k} g(k) dk \quad ,$$
$$\int \psi^*(x) x^2 \psi(x) dx = -\int g^*(k) \frac{\partial^2}{\partial k^2} g(k) dk \quad .$$

These relations imply that, for momentum-space wavefunctions, the correct position operator is $i\partial/\partial k$.

2. Relations associated with the momentum-space wavefunction looks a lot simpler when we use the wavenumber k as the variable. But, most of the books use the momentum $p = \hbar k$ as the variable and therefore define the momentum-space wavefunction as a function of p. In this problem, you will see the relation between these approaches. The momentum space wavefunction ϕ as a function of p is defined as

$$\phi(p) = \frac{1}{\sqrt{\hbar}} g(k)$$

where $p = \hbar k$ and $g(k)$ is defined in Eq. (1).

(a) Show that if $|g(k)|^2$ has the dimensions of 1/wavenumber, then $|\phi(p)|^2$ has the dimensions of 1/momentum.

(b) Show that

$$\int |\phi(p)|^2 dp = 1 \quad \Leftrightarrow \quad \int |g(k)|^2 dk = 1 \quad .$$

(c) Show that

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \phi(p) e^{ipx/\hbar} dp \quad ,$$
$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ipx/\hbar} dx \quad .$$

(d) Show that

$$\int \psi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) dx = \int \phi^*(p) p \phi(p) dp \quad ,$$
$$\int \psi^*(x) x \psi(x) dx = \int \phi^*(p) i\hbar \frac{\partial}{\partial p} \phi(p) dp \quad .$$

3. A particle has the wavefunction $\psi(x)$ given by

$$\psi(x) = N \exp(-|x|/a) \quad ,$$

where a is a constant with the dimensions of length and N is a normalization constant.

(a) Find a value for N so that ψ is normalized.

(b) Compute the momentum-space wavefunction $g(k)$.

(c) Find $\langle x \rangle$ and $\langle p \rangle$.

(d) Find $\langle x^2 \rangle$ and $\langle p^2 \rangle$.

(e) Compute $\Delta x \Delta p$ and show that the uncertainty relation is satisfied.