CALCULUS

TRIGONOMETRIC SUBSTITUTION

CASE #3: Integrals involving $\sqrt{x^2-a^2}$ (where $a>0$) by using substitution $x=a\sec\theta$.

The inverse secant substitution

Integrals involving $\sqrt{x^2-a^2}$ (where $a>0$) can frequently be simplified by using the substitution

$$x = a\sec\theta \quad \text{or, equivalently,} \quad \theta = \sec^{-1}\frac{x}{a}.$$

We must be more careful with this substitution. Although

$$\sqrt{x^2-a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|,$$

we cannot always drop the absolute value from the tangent. Observe that $\sqrt{x^2-a^2}$ makes sense for $x \ge a$ and for $x \le -a$.

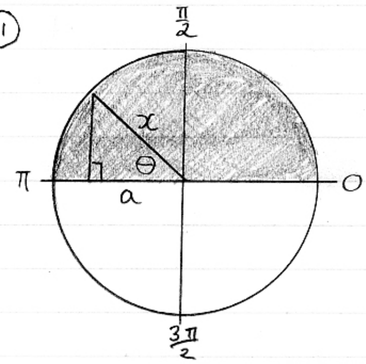
$$\text{If } x \ge a, \text{ then } 0 \le \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} < \frac{\pi}{2}, \text{ and } \tan\theta \ge 0.$$
$$\text{If } x \le -a, \text{ then } \frac{\pi}{2} < \theta = \sec^{-1}\frac{x}{a} = \arccos\frac{a}{x} \le \pi, \text{ and } \tan\theta \le 0.$$

In the first case $\sqrt{x^2-a^2} = a\tan\theta$; in the second case $\sqrt{x^2-a^2} = -a\tan\theta$.

Please see Page #2 for Questions . . .

QUESTIONS:

①



UNIT CIRCLE

Figure of $\theta = \sec^{-1}\frac{x}{a}$.

$$x = a\sec\theta \rightarrow \theta = \operatorname{arcsec}\frac{x}{a} = \sec^{-1}\frac{x}{a}$$

$$\sqrt{x^2 - a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|$$

Observe that $\sqrt{x^2 - a^2}$ only makes sence for $x \geq a$ and for $x \leq -a$.

However, how is it possible for $x \leq -a$ in the context of the unit circle. I'm not understanding something.

② Range of $\theta = \sec^{-1}\square$ is $\{0 \leq \theta < \frac{\pi}{2}\} \cup \{\frac{\pi}{2} < \theta \leq \pi\}$.

However, for the purpose of trigonometric substitution some textbooks restrict the range to $\{0 \leq \theta < \frac{\pi}{2}\} \cup \{\pi \leq \theta < \frac{3\pi}{2}\}$.

What's going on? How do justify restricting the range to this? The range $\{\pi \leq \theta < \frac{3\pi}{2}\}$ isn't even in the range of $\theta = \sec^{-1}\square$!