2200 Spring 2005. Fundamental Boundedness Results

We have stated the following results about boundedness today, and sketched one proof.

Definition: An interval (a,b) or [a,b] or (a,b] or [a,b) is called <u>bounded</u>, if and only if both a and b are finite real numbers.

Thus the only unbounded intervals are of form (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $(-\infty,\infty)$.

Definition: A real valued function f with domain the interval I, is <u>bounded on I</u> if and only if the set of values f(x) takes on, as x runs over I, is contained in some bounded interval; mopre precisely, if there are some numbers L, K such that for all x in I, $L \leq f(x) \leq K$.

I.e. the <u>values</u> of f, do not become infinitely big, either positively or negatively, as x runs all over I, even if I itself may be unbounded.

Examples:
i) The function cos is bounded for $I = (-\infty,\infty)$, since the values cos(x) all lie in the bounded interval [-1,1]. I.e. it is the y values that matter, not the x values, at least here in the definition of boundedness.

Thus cos is bounded on any smaller interval I on which we consider it as well.

ii) The function x^2 is unbounded on the interval $I = [0,\infty)$, since the values x^2 become infinitely large as x gets large positively.

iii) The function x^2 is bounded on the interval [-100,100], since for x's in that interval x^2 always lies in the bounded interval $[0, 10^4]$.

iv) The function tan is unbounded on the interval $(-\pi/2, \pi/2)$, since $\tan(x) = \sin(x)/\cos(x)$ becomes infinitely large as x approaches $\pi/2$, since $\sin(\pi/2) = 1$, and $\cos(\pi/2) = 0$. In particular tan(x) does not have a finite limit as x approaches $\pi/2$.

v) The function $f(x) = \tan(x)$ for x in $(-\pi/2,\pi/2)$ and $f(\pi/2) = 0 = f(-\pi/2)$, is also unbounded on the interval $[-\pi/2, \pi/2]$, but is not continuous at the end points of the interval.

Example iv) shows that the set of <u>values</u> of a continuous function may be unbounded even when the <u>domain</u> is bounded. I.e. even when the x's remain bounded, the f(x)'s can become unbounded. The domain interval is however open in this example. Example v) shows a discontinuous function can also be unbounded on a closed bounded domain interval.

We want to give a set of criteria that describe one important situation when the set of values of a function is bounded.

Theorem: If f is a real valued function on a domain interval I such that:

1) I is both closed and bounded, i.e. I = [a,b] where a and b are finite real numbers, and
2) f is continuous everywhere on I = [a,b],

then f is bounded on [a,b].

I.e. then there exists a positive number K such that for every x in [a,b], we have $-K \leq f(x) \leq K$.

Remark:
This shows that unboundedness of a function can only be produced in one of the following ways, (as in the previous examples):
i) on an unbounded domain interval, say $I = (a,\infty)$,
ii) on an open ended bounded domain interval, say (a,b),
iii) or by a discontinuous function, (this can occur on any interval).

Here is my "proof" of the big theorem above: the proof proceeds by contradiction. I.e. I will show that if a function f is unbounded on a closed bounded domain interval, then the function f must be discontinuous at some point of that interval. I will give the proof only for a special case, where say I = [0,1].

First a preliminary result:
Lemma: If f is continuous at a, then f is bounded on some interval containing a.
proof: This is immediate from the definition of continuity. I.e. if f(a) = c say, then there is some interval (a-d,a+d) on which f changes by less than 1. I.e. for all x in the open interval (a-d,a+d), f(x) lies between f(a)-1 and f(a)+1. Thus f is bounded on the interval (a-d,a+d). **QED.**

Remark: Intuitively, this says of f is finite at a, and if for x near a, we have f(x)is close to f(a), then f cannot get infinitely big for x near a.
If we say the same thing backwards, we get:

Corollary: If f is unbounded on every interval containing a, then f cannot be continuous at a, indeed f cannot even have a finite limit at a.
proof: same argument. **QED.**

OK now we are ready to prove the theorem. What we will prove is, if f is unbounded on the interval [0,1], then there is some point in that interval at which f is not continuous. When you say this backwards, it proves that if f is continuous at every point of [0,1], then f must be bounded on that interval.

Now we are going to construct a real number by giving its decimal expansion. Since the decimal expansion of a real number is usually infinite, we cannot actually write down all the entries but we will tell a rule for finding them, one at a time.

OK, assume f is unbounded on the interval [0,1]. Then subdivide the interval into ten smaller intervals [0,.1], [.1,.2],.....,[.8,.9], [.9,1], and ask whether f is unbounded on any of these. Well it must be, because if f were bounded on each one of these intervals, there would be ten different bounds for f on these intervals, but we could just take the largest of those ten bounds and that would be a bound for f on the whole interval.

I.e. since there is no bound for f on the whole interval, there must be some smaller interval on which there also is no bound. Indeed there might be more than one, but there must be at least one. Suppose then that f is unbounded say on the interval [.2,.3]. Then the number we are looking for starts out as .2.

Now subdivide this smaller interval again into ten smaller intervals,

i.e. into the intervals [.2.21]. [.21,.22], [.22,.23],.....,[.28, .29], [.29, .3].

Now again there must be at least one of these intervals on which f is unbounded for the same reason as before. Say f is unbounded on [.24, .25]. Then take as the next part of the decimal expansion of our desired number .24.

Keep going forever, and we produce an infinite decimal expansion, say c = .2467911530548..... , i.e. a real number c in our interval [0,1]. Now f was unbounded on every interval whose left end point was a finite part of this decimal, and whose right end point was the next larger roundup with the same beginning terms. If we look at the first n terms of the decimal expansion, that interval had length only 1/(10)^n, and contained our number c, and also f was unbounded on that interval.

Since f is unbounded on a sequence of intervals, all containing c, and having lengths approaching zero, it follows that f cannot be not bounded on any finite interval containing c. I.e. if f were bounded on some finite interval containing c, then eventually any number with the same first n terms as c would lie in that interval, but we have seen that f is not bounded on any interval of numbers with the same first n terms as c. I.e. since f is not bounded on any interval of form (c - 1/(10)^n , c+ 1/(10)^n), f cannot be bounded on any interval containing c. Thus by our earlier remarks, f cannot be continuous at c. But one can see, I hope, that c does belong to the interval [0,1]. Hence f was not continuous on that interval after all.

Thus if any function f is continuous at all points of [0,1], then f is actually bounded on [0,1].
QED

Remark: This is considered to be the hardest theorem in the whole subject, so if you got some feeling for why it is true, good for you!