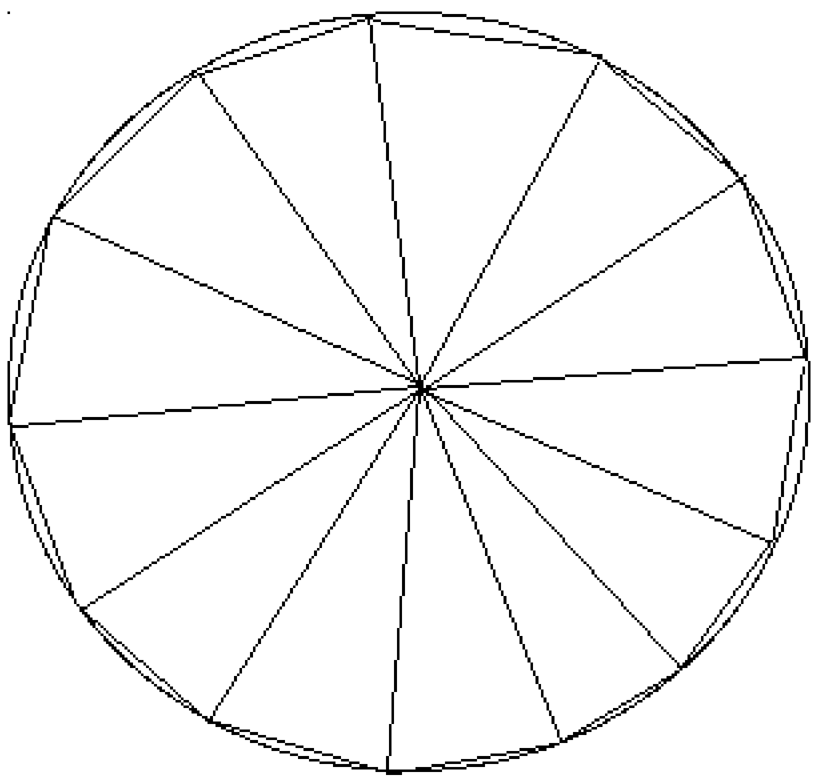
2210 day1:

Archimedes and the problem of length, area, and volume

Almost 2000 years ago the great mathematician Archimedes pondered questions such as how long is a circle? how much area does it contain? how much water does it take to fill a conical tank? His answers to these questions are famous to this day, and his methods laid the groundwork for the integral calculus discovered by Newton and Leibniz in the 19th century. We want to begin by re examining some of Archimedes work.

The first question was how long is a circle? Euclid knew the answer depends only on the radius, i.e. there is some number called π, such that any circle of radius r has length $2\pi r$. I.e. the larger the radius, the longer the circumference, and the relationship is "linear", i.e. the circumference is "proportional" to the radius.

But just what is the proportionality factor π? Archimedes tackled this by the method of approximation, i.e. of limits. He reasoned that a circle was like a regular polygon, with infinitely many extremely short sides. Thus a polygon with a large number of sides should be a good approximation to a circle, and hence its circumference should be a good approximation to the circumference of a circle.



Thus in the picture above, the sum of the lengths of the bases of the triangles making up the polygon gives a good approximation to the circumference of the circle, and the more sides the polygon has, i.e. the more triangles we use, the better the approximation. I am not going to do the details, but by carrying out this sort of computation, Archimedes gave some very good approximations to π. In particular the one we often use 22/7 was found by him. He also found much better ones, but I do not remember them.

The point I want to make is that there are two steps to this problem.
Step 1: Describe the quantity you want to measure as a limit.
Step 2: Make calculations to approximate this limit.

In this case, the answer to step 1, is that:

The circumference of a circle is equal in length to the limit of the circumferences of inscribed regular polygons, as the number of sides of the polygons approaches infinity.

This is a complicated description of the number to be found, but it is precise, and once we have this precise description, we can approximate the circumference as closely as we like, by using more and more triangles.

But things get better.

Wonderful surprise:
The miraculous thing that we also want to study in this course, is that there is a third step that is sometimes possible.

3. Figure out a clever way to guess, and then prove, the exact value of the limit.

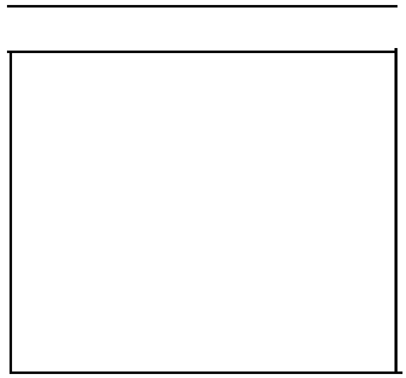
We have no way to do this for the length of the circle, but let's try it on another of Archimedes problems, the area of a circle.

Using the same principle as above, the area of the circle should be the limit of the areas of those inscribed polygons, which can be computed by summing up the areas of the triangles making up the polygons.

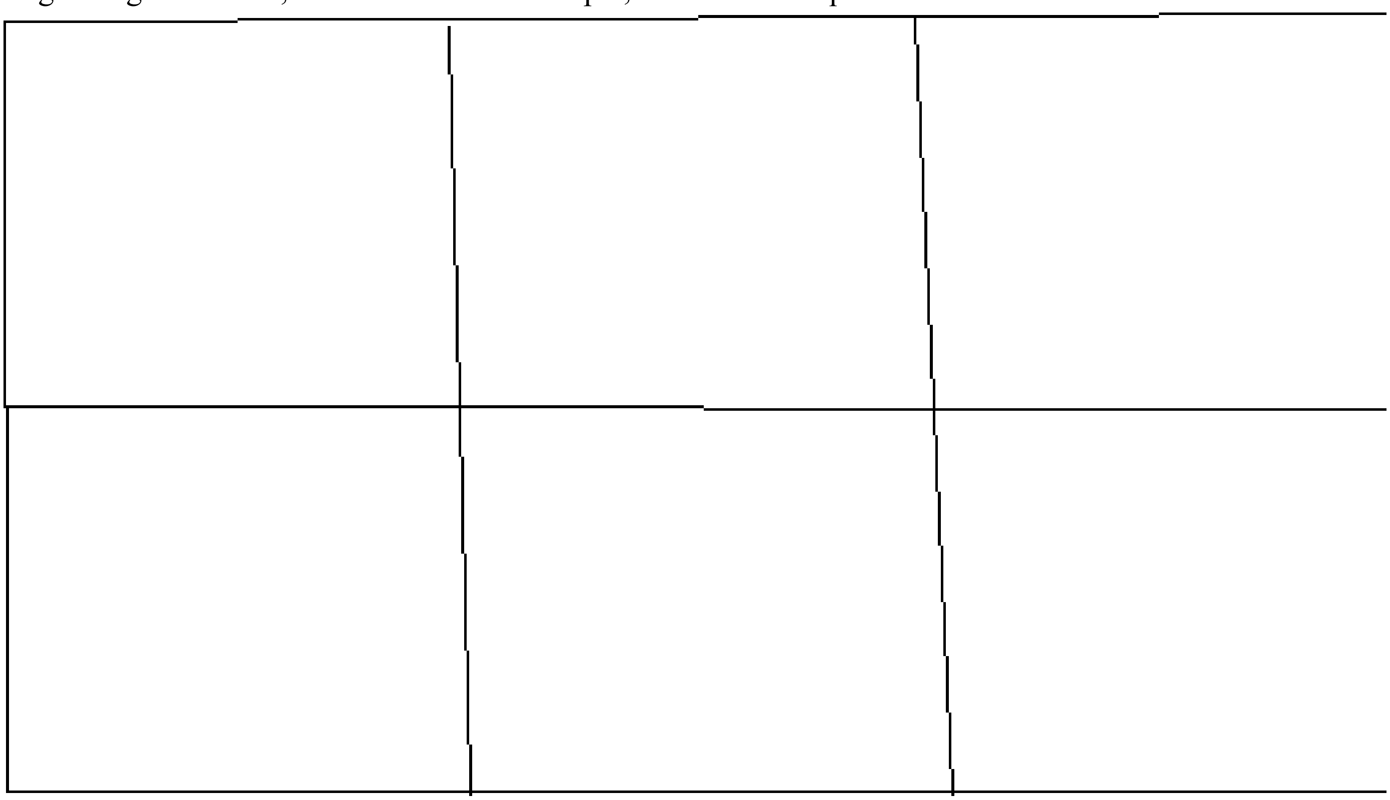
This time we are going to try to give an "exact" answer to the question: "What is the area of the circle?"

First we recall the formula for the area of a triangle. By the “area” of a plane figure, we mean the number of square units that "fit into" the figure.

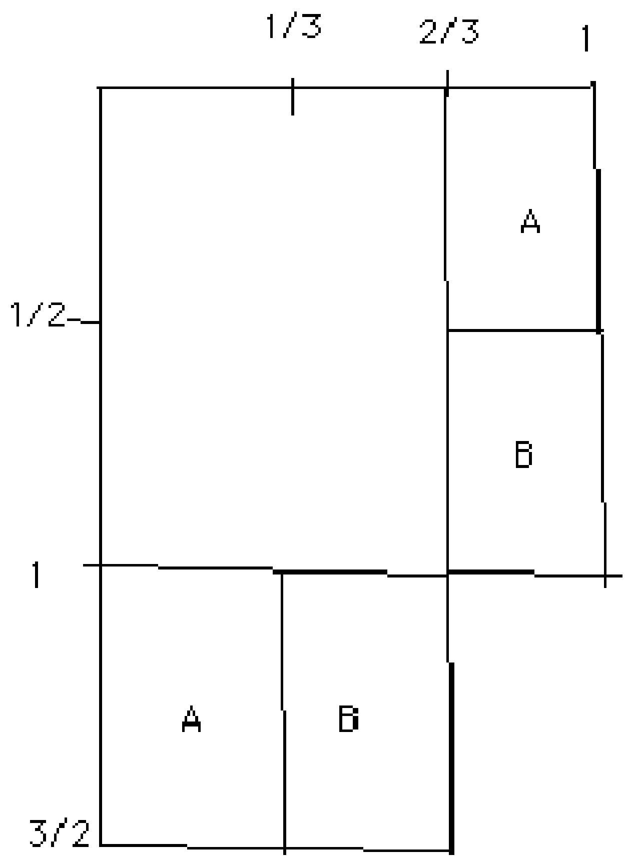
So choose a unit of length, say the segment below, and form a square with sides of that length, as below. Then that square is our unit of area. I.e. a choice of unit of length determines a



corresponding unit of area.
It seems fairly clear that a rectangle whose sides have integer length a, and b, can be exactly filled up by atimesb = ab of our unit squares. So we say that a rectangle whose sides have integer lengths a and b, has area ab. For example, here 2x3 = 6 square units of area.

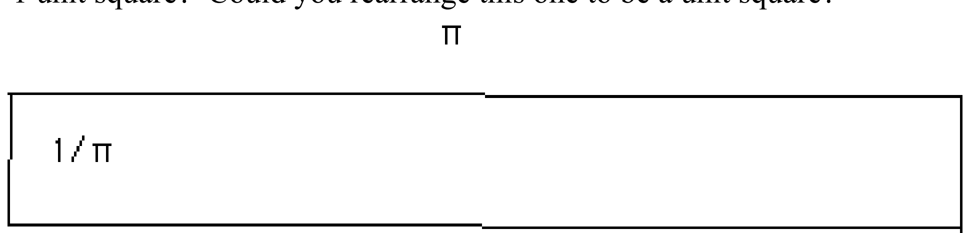


It is not as obvious that a rectangle whose sides have length 3/2 unit, and 2/3 unit should also have (3/2)(2/3) = 1 square units of area, but this works too. In the example below we show how to rearrange a rectangle measuring (2/3) by (3/2) to get a unit square.

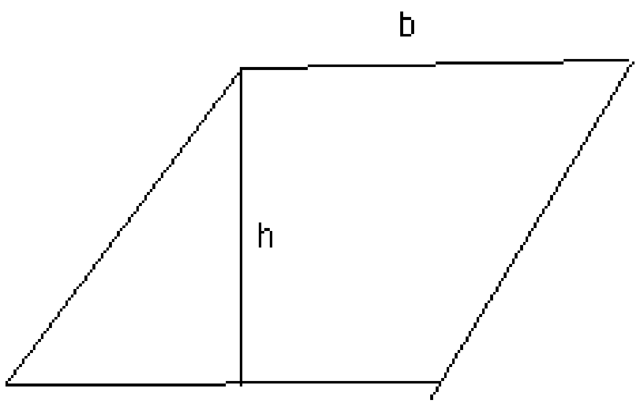


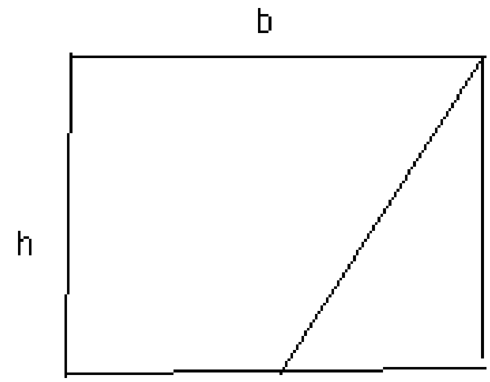
So we accept that the area of any rectangle with sides of lengths X and Y, has area XY square units, the usual formula for area of a rectangle, even though I wager most of us have never seen a proof that this is correct for all non integer values of the sides.

I.e. what does it mean to say that a rectangle whose sides have length π and $1/\pi$, contains $\pi(1/\pi) = 1$ unit square? Could you rearrange this one to be a unit square?



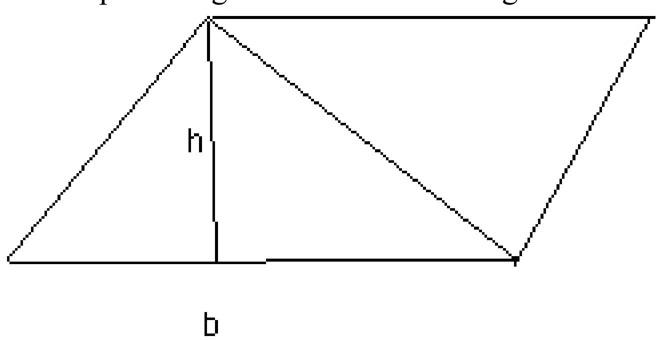
Once we know how to find areas of rectangles, then we can find the area of a parallelogram by rearranging it to be a rectangle as usual.





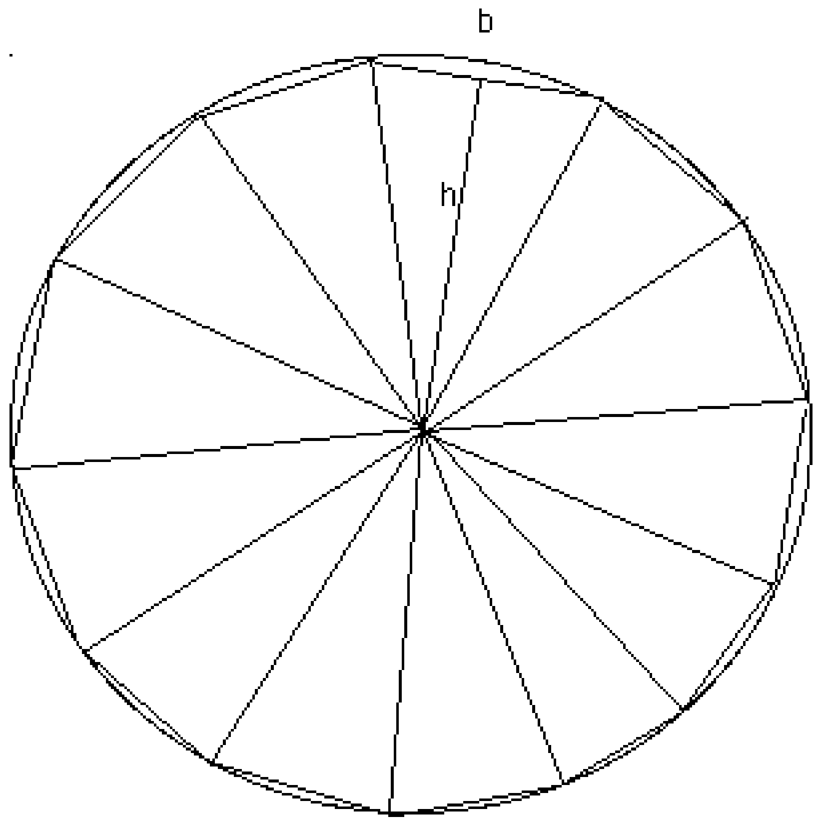
So a parallelogram with height h and base b also has area bh, like a rectangle.

Now a triangle of base b and height h has area (1/2)bh, as we can see by combining two of them to make a parallelogram of base b and height h.



So now we know the area of a triangle of base b and height h is (bh/2).

Now we know everything Archimedes knew when he found the area of a circle, so we should be able to do that too. Look back at the circle approximated by inscribed polygons.



Now just write down the sum of the areas of these triangles. Say there are N of them. Then each triangle has area bh/2, and there are N triangles so their areas add up to A(N) = Nbh/2. Now let the number N of triangles increase to infinity, and ask what is the limit of this number Nbh/2. We know Nb is the sum of the bases of the triangles, i.e. the circumference of the polygon approximating the circle, so as N approaches infinity, Nb approaches the circumference C of the circle. (Of course b = the abse of one triangle, so b is getting shorter as we use more triangles.) Moreover the height h of each triangle approaches the radius of the circle. So this number A(N) = (Nb)h/2approaches the number A = Cr/2 = half the product of the radius and the circumference of the circle.

Now since we know the circumference C equals $2\pi r$, this limit is
$A = Cr/2 = (2\pi r)r/2 = \pi r^2$, the answer we were expecting.

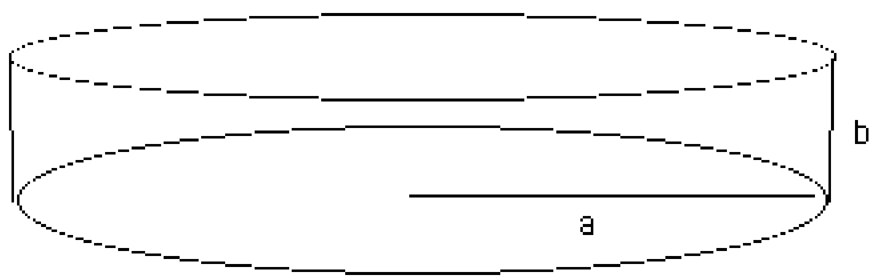
Thus if we know the circumference of the circle, we can compute exactly the area of the circle as a limit!

Now just as area is measured in square units, volume is measured in cubic units, where a unit cube is a cube whose sides are all unit length. then by drawing some pictures you can convince yourself that the volume of a rectangular block equals the area of the base times the height. I.e. if the block has a base which is a rectangle measuring 2 by 4, there are 8 squares in the base. Assume the height of the block is 3. Then on each of the 8 squares in the base we erect a stack of 3 unit cubes. So all together we get 3(8) = 24 unit cubes of voloume. Another way to look at it is like a building 3 stories high. The first story is composed of 8 unit cubes. Each of the three stories has 8 unit cubes and these are piled on top of each other to make a building 3 stories high.

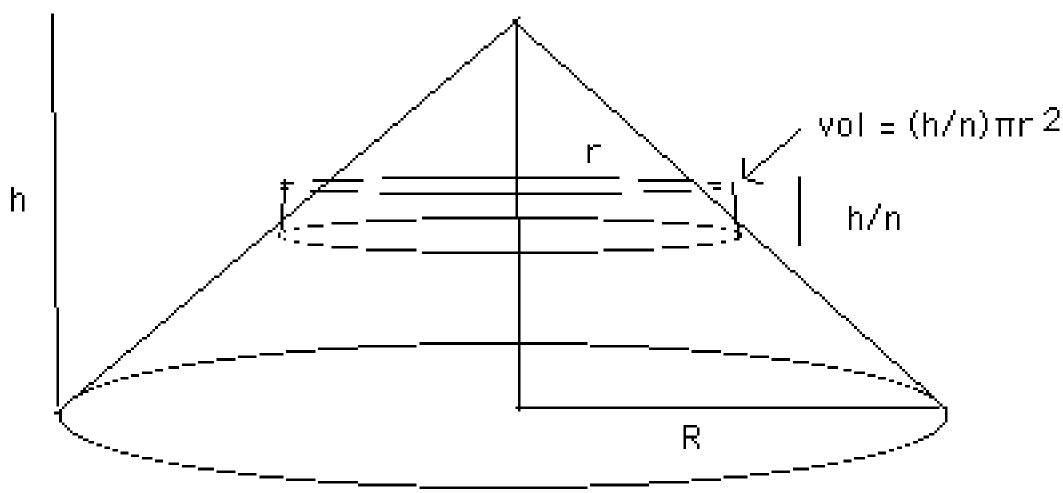
Thus each story contains 8 unit cubes and there are 3 stories so we get a total volume in our building of 24 unit cubes. Thus the volume of a block equals the area of the base times the height.

Now what if we have a block with a circular base like a wheel of cheese. I.e. a circular cylinder with straight sides and a circular base. Then it is still true that the volume equals the area of the abse times the height. So the volume of any cylinder, i.e. any solid figure with vertical sides, equals the area of the base times the height.

In particular, we are going to assume the volume of a circular cylinder is the product of the area of the base times the height of the cylinder. Thus the circular cylinder below has volume $\pi a^2 b$.



Now let's try another problem solved by Archimedes, the volume of a circular cone. This time we will approximate the cone by a pile of smaller and smaller cylinders, like a pile of washers. We only draw one of the washers below. We have divided the height into n equal parts, and the washer we show has height h/n. To find its radius r we use the principle of proportionality. I.e. if this washer is the 3rd one say from the top, then it has a distance (3/n)h from the top. So similar triangles tells us that (3/n)h/h = r/R. I.e. 3/n = r/R, so r = 3R/n.



Similarly if this is the "jth" washer, then we get r = jR/n. Hence the volume of the jth washer is $\pi r^2(h/n) = \pi(jR/n)^2(h/n) = \pi R^2 j^2 h/n^3$.

Thus if we add up the volumes of all n of the washers, i.e. add these formulas for j = 1,....,n, we get

$\sum_{j=1...n} \pi R^2 j^2 h/n^3 = (\pi R^2 h/n^3)(\sum_{j=1...n} j^2)$. There is a formula on p.321 for the sum of the first n squares, and it is
$(\sum_{j=1...n} j^2) = n(n+1)(2n+1)/6$.

Plugging this in gives the approximate volume of $[n(n+1)(2n+1)/6](\pi R^2 h/n^3)$. Let's simplify this a little. Multiplying out the formula $n(n+1)(2n+1)/6$ gives $2n^3/6 + q(n) = n^3/3 + q(n)$, where q is quadratic in n.

Thus we get approximate volume = $(\pi R^2 h/3) + (\pi R^2 h\, q(n))/n^3$. Now if we let n approach infinity, the fraction $q(n)/n^3$ approaches zero, and we get the limiting volume as

vol = $\pi R^2 h/3$, for a cone of base radius R, and height h.

So this is how Archimedes found the volume of a cone. We will use his limiting method to describe more complicated areas and volumes, and then we will take advantage of the fact that we also know differential calculus, to find an easer way to compute the limits that come up. So please review derivatives, their definition, their computations rules, and their various geometric and other interpretations (slope, rate of change, velocity, etc..).

HW: Work back through this computation for the volume of a cone, until you understand it, and then imitate it to find the volume of a hemisphere. The answer is supposed to be $2\pi R^3/3$ (for a half sphere of radius R).

Also
Read section 5.3, and work as many examples and problems as you can,
especially example 7, p.324.

Notice the useful formulas on page 321, formulas (7),(8),(9).

Then work out my problems assigned,

{37, 41, 46, 47, and the volume of a hemisphere problem}

problem 37 should resemble in some ways the volume problems we have been doing. Why?

Then repeat "Archimedes greatest achievement":

Prove that a sphere that fits exactly inside a cylinder takes up precisely 2/3 of the volume of the cylinder.

That homework is to turn in on Friday! But you should be ready to put the hemisphere problem on the board wednesday.

For those of you who are already lost, go back and review limits. like the limit of $(4n^3 + 6n^2 + n)/n^3$ as n goes to infinity.

It is no crime to go back and take or audit math 2200 instead of this course if you need it.