Math 2300H. Limits, continuity, and derivatives

The basic concept is that of a continuous function. Intuitively a function f is continuous at a point a, if the values f(x) for $x \neq a$ but x near a, give good approximations to the value f(a). But how good is good? Within 1/1,000,000 is not near enough! We want to be able to approximate f(a) by f(x) to within ANY degree of accuracy, just by taking x near enough to a. The precise definition, arrived at after many years of experimentaion and refinement, is this:

Definition: f is continuous at a if and only if, for every number $e>0$, there is a number $\partial > 0$, such that whenever x is in Domain(f) and $|x-a| < \partial$, then $|f(x) - f(a)| < e$.

For example, $f(x) = 9x$ is continuous at $x=3$ because if e is a positive number and we take $\partial = e/9$, then ∂ is positive, and for every x in Dom(f), if $|x-3| < \partial = e/9$, then multiplying this inequality through by 9 shows that $|9x-27| < e$, where $f(x) = 9x$ and $f(3) = 27$.

As another example, $f(x) = x^2$ is continuous at $x = 1$, because if $e > 0$, and we choose $\partial = \min\{ e/3, 1\}$ then $|x-1| < \partial$ implies $|x| - |1| \leq |x-1| < 1$, so $|x| < 1 + |1| = 2$, hence $|x+1| < |x| + 1 < 3$. Thus $|f(x) - f(1)| = |x^2-1^2| = |x-1||x+1| < (e/3)(3) = e$. QED.

As another example, cos(ø) is continuous at ø = 0. Recall that if ø is a small number, to compute cos(ø) (in theory) we start from (1,0) and draw an arc along the unit circle of arclength = ø, counterclockwise if ø > 0 and clockwise if ø < 0, and then cos(ø) is the x coordinate of the endpoint p = (cos(ø), sin(ø)) of the arc. Thus we have a small "circular right triangle" with base the segment joining (cos(ø),0) to (1,0) on the x axis, and 3rd vertex at p = (cos(ø), sin(ø)) on the unit circle, and with "hypotenuse" the circular arc joining (1,0) to p.

(Draw a picture.) Then |ø| = the length of the circular hypotenuse which is even longer than a straight hypotenuse would be for the triangle with these vertices, which is longer than the base side having length |cos(ø)-1|. Thus given a small positive number $e > 0$, we can choose $\partial = e$. Then if $|ø-0| < \partial$, we have $|\cos(ø) - \cos(0)| = |\cos(ø) - 1|$ = length of base < length of "hypotenuse" = $|ø| < \partial = e$. Thus cos is continuous at 0.

Intuitively, addition and multiplication are continuous operations, in the sense that if x is near a and y is near b then x+y is near a+b and xy is near ab. This intuitive perception has a precise formulation as follows:

1) Theorem: If f and g are both continuous at a, so are f+g and fg.
We will prove this in a minute. First here are some more properties of continuity:
2) If f and g are continuous at a, and if $g(a) \neq 0$, then f/g is continuous at a.
3) If f is any polynomial with real coefficients, then f is continuous at every real number a.
4)(i) If n is any odd positive integer, then the nth root function $x^{1/n}$ is continuous at every real number a.
(ii) If n is any even positive integer, then the nth root function $x^{1/n}$ is continuous at every non negative real number $a \geq 0$.
5) cos and sin are both continuous at every real number ø.
6) If $b > 0$ is any positive real number, then

(i) the exponential function $f(x) = b^x$ is continuous at every real number a.
(ii) the log function $g(x) = \log_b(x)$ is is continuous at every positive real number $a > 0$.
proof of Theorem 1): (for f+g): let $e > 0$ be given. We must find $\partial > 0$ so that $|x-a| < \partial$ implies $|(f+g)(x) - (f+g)(a)| < e$. Since f is continuous at a, if we take $e_1 = e/2$, then we can choose $\partial_1 > 0$ so that if $|x-a| < \partial_1$ then $|f(x) - f(a)| < e_1 = e/2$. Similarly since g is continuous at a, if we take $e_2 = e/2$, we can find $\partial_2 > 0$ so that $|x-a| < \partial_2$ implies $|g(x) - g(a)| < e_2 = e/2$. Then taking $\partial = \min\{\partial_1, \partial_2\} > 0$, we get that if $|x-a| < \partial$ then $|x-a| < \partial_1$ and $|x-a| < \partial_2$ so $|(f+g)(x) - (f+g)(a)| =$
$|f(x) - f(a) + g(x) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < e_1 + e_2 = e/2 + e/2 = e$. QED.

proof of continuity for fg: We will break the desired quantity into three pieces and make each one less than e/3.) Note that $f(x) = f(a) + (f(x)-f(a))$, and $g(x) = g(a) + (g(x)-g(a))$, so
$f(x)g(x) = f(a)g(a) + f(a)(g(x)-g(a)) + g(a)(f(x)-f(a)) + (f(x)-f(a))(g(x)-g(a))$,
hence $f(x)g(x) - f(a)g(a) = f(a)(g(x)-g(a)) + g(a)(f(x)-f(a)) + (f(x)-f(a))(g(x)-g(a))$.

Clearly we can make all three terms on the right as small as desired by making $(f(x)-f(a))$ and $(g(x)-g(a))$ small enough. More precisely, if $g(a) \neq 0$ choose $e_1 = \min\{1, e/3, e/(3|g(a)|)\}$, and if $g(a) = 0$, choose $e_1 = \min\{1, e/3\}$. Then since f is continuous at a, choose $\partial_1 > 0$ so that $|x-a| < \partial_1$ implies $|f(x) - f(a)| < e_1$. Thus $|x-a| < \partial_1$ implies the second term $|g(a)(f(x)-f(a))| < |g(a)|.(e/(3|g(a)|) = e/3$.

Similarly, if $f(a) \neq 0$, choose $e_2 = \min\{1, e/3, e/(3|f(a)|)\}$ and if $f(a) = 0$, choose $e_2 = \min\{1, e/3\}$. Then choose $\partial_2 > 0$ so that if $|x-a| < \partial_2$ then $|g(x)-g(a)| < e_2$. Then $|x-a| < \partial_1$ implies the first term $|f(a)(g(x)-g(a))| < |f(a)| (e/(3|f(a)|) = e/3$.

Finally choose $\partial = \min\{\partial_1, \partial_2\} > 0$. Then $|x-a| < \partial$ implies the third term $|(f(x)-f(a))(g(x)-g(a))| < 1(e/3) = e/3$.

Thus $|x-a| < \partial$ implies $|f(x)g(x) - f(a)g(a)| = | f(a)(g(x)-g(a)) + g(a)(f(x)-f(a)) + (f(x)-f(a))(g(x)-g(a))|$

$\leq | f(a)(g(x)-g(a))| + |g(a)(f(x)-f(a))| + |(f(x)-f(a))(g(x)-g(a))| < e/3 + e/3 + (e/3) = e$. QED.

LIMITS
The concept of limit is primarily of interest when we have a function that is not continuous at a, and we ask whether it is possible to define it or redefine it at a so that it becomes continuous there.

Since f may already be defined at a but not continuous, we want to find the value f <u>should</u> have, not the value it <u>does</u> have, so we have to define this limit without looking at the actual value f has at a. After we find the limit to be L say, and perhaps then learn that in fact L does equal f(a), then we know that f was already continuous at a. On the other hand if we already know f is continuous at a, then we do not have to do any work to find the limit since then we know the limit is just f(a).

Intuitively the limit L = $\lim_{x \to a} f(x)$ is the number which is being approximated well by the values f(x) for x ≠ a and x near a. The precise definition is this:

Definition: If f is defined on some deleted neighborhood of a, (possibly also at a but not necessarily so), then $\lim_{x \to a} f(x) =$ L if and only if for every e > 0, there is some $\partial > 0$ such that if x is in Dom(f) and 0 < |x-a| < ∂, then |f(x)-L| < e.

NOTICE: The "0 < |x-a|" part of the definition means that x is never allowed to equal a, in defining the limit. Thus the value of f at a plays no role in defining the limit $\lim_{x \to a} f(x)$. This is why I like to write $\lim_{\substack{x \to a \\ x \neq a}} f(x)$ for the limit, to remind us of this fact, but it takes more chalk and more time.

If there is no such number L, we say f has no limit at a, or the limit of f as x approaches a does not exist, or some such phrase.

It follows from these precise definitions that the intuitive meaning given above of limit, as the value that makes f continuous is justified. I.e. one can <u>prove</u> from these definitions, that if f is defined on a neighborhood of a, then f is continuous at a if and only if:

(1) $\lim_{\substack{x \to a \\ x \neq a}} f(x)$ exists, and

(2) $\lim_{\substack{x \to a \\ x \neq a}} f(x)$ = f(a).

This gives us the world`s easiest way to find certain limits:
(I) Principle of continuity: If f is defined on a neighborhood of a and continuous at a, then $\lim_{\substack{x \to a \\ x \neq a}} f(x)$ = f(a), in this case (and only in this case!) to find the limit just take the value at a.

Example: $\lim_{x \to 0} \cos(x) = \cos(0) = 1$.
Example: $\lim_{x \to 9} x^{1/2} = 9^{1/2} = 3$.
Example: $\lim_{x \to 2} x^3 = 2^3 = 8$,.

Of course we (or rather, Isaac Newton) would never have invented the concept of limit only to use it in such trivial situations. We will need more tools for most examples. Amazingly however, a tiny alteration of this principle gives us a really useful method for finding limits as follows.

The deleted neighborhood principle: If two functions f and g are both defined, and both have the same values, on some <u>deleted</u> neighborhood of a, then they have the same limit as x approaches a. I.e. if f has a limit as x approaches a, then g does too, and they have the same limit. This is true whether or not they are defined at a, and if they are defined at a, it is true whether or not the values f(a) and g(a) are equal.

Putting the two previous principles together gives us a very useful method for finding limits:
The principle of continuous extension: If f,g are two functions defined and having the same values on some deleted neighborhood of a, and if also g is defined and continuous at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ = g(a).

This is true regardless of the value f has at a, or whether f is even defined at a. The function g is called the "continuous extension" of f to the point a, which explains the name of the principle.

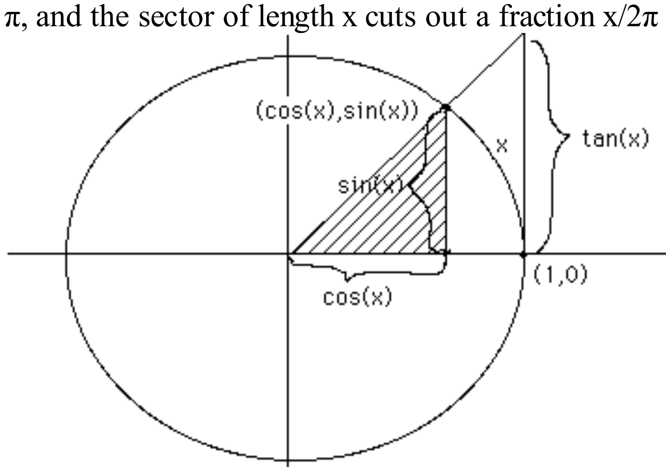
Example: If f(x) = $\frac{x^2-4}{x-2}$, then f is defined except at x=2, and if g(x) = x+2, then g(x) = f(x) except for x=2, while g is continuous at 2. Hence $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exists and equals $\lim_{x\to 2}(x+2)$ = 4. I.e. x+2 is the continuous extension of $\frac{x^2-4}{x-2}$.

Example: If f(x) = $\frac{x^{1/3}-2}{x-8}$, then f is defined except at x = 8, but x-8 = $(x^{1/3}-2)(x^{2/3} + 2x^{1/3} + 4)$, so if g(x) = $1/(x^{2/3} + 2x^{1/3} + 4)$, then g is the continuous extension of f to x=8. Thus $\lim_{x\to a} f(x)$ = $\lim_{x\to a} g(x)$ = $1/(8^{2/3} + 2.8^{1/3} + 4)$ = 1/(4+4+4) = 1/(12).

Even this trick does not help us if say f(x) = sin(x)/x since we do not yet know how to guess a continuous extension of this f to x=0, and do not even know if there is one. So we need one more useful principle.

The squeeze principle: If f, g, h are all defined, and if g(x) ≤ f(x) ≤ h(x) for all x in some deleted neighborhood of a, and if $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ not only both exist but are both equal, say to L, then $\lim_{x\to a} f(x)$ also exists and is also equal to L.

The best case here is if both g,h are continuous at a. For example, if we compare the areas of the two triangles and the circular sector in the picture below, and use the fact that the area of a unit circle is π, and the sector of length x cuts out a fraction x/2π of the whole circle,



then we get the inequality $(\sin(x)\cos(x))/2 \leq x/2 \leq \tan(x)/2 = \sin(x)/2\cos(x)$. Hence half the reciprocals satisfy the inequality $\cos(x)/\sin(x) \leq 1/x \leq 1/(\sin(x)\cos(x))$, and if we multiply this through by $\sin(x)$ we get the inequality $\cos(x) \leq \sin(x)/x \leq 1/\cos(x)$, for x near 0.

If you did not follow the derivation just take this inequality on faith. Now use the squeeze principle on this. I.e. the two end functions are both continuous at x = 0 so they have limits there which we can compute by evaluating them at x = 0. But when we do, we get the same answer, namely 1, so the middle function, our old mystery function $\sin(x)/x$, also has a limit as x approaches 0, and that limit is also 1. I.e. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$.
(Actually we assumed in our picture that x was positive so we only got the "right hand limit" of this function, but using the fact that $\sin(-x)/(-x) = \sin(x)/x$ we can do the case of negative x also and we get the same answer.)

We will use limits in this course especially to find slopes of tangent lines, i.e. to find "derivatives". The official definition is the following.

Definition: If f is defined on a neighborhood of a point a, we say f has a derivative at a if and only if the following limit exists (and is finite)

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Notation: If this limit exists we call it $f'(a)$ or $\dfrac{df}{dx}(a)$ or $\left.\dfrac{df}{dx}\right|_{x = a}$, or $D_x f(a)$, or something else with a 'd' or 'D' in it and an 'f' ' and an 'a'. We also see it written as $\dfrac{dy}{dx}(a)$ when $y = f(x)$. If we want to use x as the letter for the point where we are taking the derivative, then we define the limit as

$\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ and write the limit as $f'(x)$ or sometimes as $\dfrac{df}{dx}$ or just $\dfrac{dy}{dx}$.

The number $f'(a)$ is called the derivative of f at a, (and $f'(x)$ is called the derivative of f at x).

Definition: The graph $y = f(x)$ has a tangent line at the point $(a,f(a))$ if and only if $f'(a)$ exists. If $f'(a)$ does exist then the tangent line is defined to be the unique line through $(a,f(a))$ with slope $f'(a)$.

Thus the tangent line has equation $y-f(a) = f'(a)(x-a)$. It will generally <u>not</u> be true that this line meets the graph only once, as happens for tangents to a circle, although this is true for convex graphs like parabolas. (Look at some tangent lines to $y = x^3$ on the other hand.)