2300H: Monday 10/2/2000.
"Combining" continuous and differentiable functions
In practice, the most efficient way to provide a large supply of continuous and differentiable functions is to prove that a few basic functions are continuous or differentiable, and then prove that the fundamental ways of combining fuinctions to form new functions preserve these properties. Most functions we study in this course are formed from:

The four basic types of functions:
(1) constants,
(2) x
(3) sin(x), cos(x)
(4) a^x, where $a > 0$,

by using:
The three fundamental processes for combining functions:
(1) algebraic operations, (adding, subtracting, multiplying and dividing),
(2) composition of functions,
(3) inversion of functions (with respect to composition).

Recall a continuous function f defined on an interval I is "invertible" if and only if f is either strictly increasing or strictly decreasing on I. In either case, we say f is strictly "monotone" on I. If f is continuous and invertible on I, then the inverse function g = "f^{-1}" is defined on the interval J of values of f. This means g(f(x)) = x for all x in I, and f(g(y)) = y for all y in J.

These few functions and operations give a huge supply of continuous and differentiable functions, because of the following three theorems:

I.Existence Theorem for continuous and differentiable functions:
All four basic types of functions are continuous everywhere, and also differentiable everywhere on the real line. (This is not as easy to prove for a^x as for the others. To be honest, it is hard even to define the functions sin(x), cos(x), and a^x.)

II. Continuity Theorem: All three fundamental operations on functions "preserve" continuity, if we avoid dividing by zero.
More precisely:
(1) If f and g are both continuous at a then so are f+g, f-g, and fg. If also $g(a) \neq 0$, then f/g is continuous at a.
(2) If f is continuous at a, and g is continuous at f(a), then g0f is continuous at a, [where (gof)(x) denotes g(f(x))].
(3) If f is continuous everywhere on I and strictly monotone there, then $g = f^{-1}$ is continuous everywhere on the interval J of values of f.

III. Differentiability theorem: All three operations on functions "preserve" differentiability, if we avoid dividing by zero. I.e.:
(1)(i) If f and g are both differentiable at a, then so are f+g, f-g, and fg. If also $g(a) \neq 0$, then f/g is differentiable at a.

(ii) With those assumptions, the following formulas for the derivatives hold:

$(f\pm g)'(a) = f'(a) \pm g'(a)$;

$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$, (product rule);

$(f/g)'(a) = \{f'(a)g(a) - f(a)g'(a)\}/g^2(a)$, (quotient rule).

As a special case of the product rule, if c is constant, $(c.f)'(x) = c.f('x)$.

(2) (chain rule) If f is differentiable at a, and g is differentiable at f(a), then gof is differentiable at a, and $(gof)'(a) = g'(f(a)).f'(a)$.

(3) (inverse function rule) If f is differentiable on an interval I, and if for every x in I, $f'(x) \neq 0$, then f is monotone on the interval I, and $g = f^{-1}$ is differentiable on the interval J of values of f. Moreover, $(f^{-1})'(y) = 1/f'(f^{-1}(y))$.

I.e. if $f(x) = y$, so that $f^{-1}(y) = x$, then $(f^{-1})'(y) = 1/f'(x)$. This why we need to assume that $f'(x)$ is never zero at points x in I, in order to take derivatives of f^{-1} at points y in J.

More ways of combining functions.

There are three more important operations on functions, which cannot always be carried out, but which frequently give very interesting new functions when they can be done:

(i) taking limits of functions,

(ii) differentiating functions,

(iii) "integrating" functions.

We know about differentiating functions to get other functions. However by our rules for derivatives, the derivative of a familiar function is also a familiar function. so you do not get really NEW functions this way. The integral of a familiar function is not always so familiar however, and this gives a great way to cook up really new functions.

For example the integral of 1/x is ln(x) the "natural log of x", the integral of $1/(1+x^2)$ is arctan(x), and the integral of $1/(1-x^2)^{1/2}$ is arcsin(x), so if we took integrals we would not even need to include either exponential or trig functions as basic functions because we could get them by integrating reciprocals of polynomials and then inverting the results. I.e. if we added one more fundamental process for combining functions, integrating, then we could get all our four basic functions just from the constants and x.

Moreover the integral of $1/(1-x^4)^{1/2}$ is so new and interesting, it was studied by the Bernoulli's, Euler, and Legendre in the 1700's, then further by Gauss, Abel, and Jacobi in the 1800's. In the mid 1800's Galois and especially Riemann generalized these ideas further to the study of functions obtained by inverting integrals of functions of higher order such as $1/(1-x^6)^{1/2}$. A deep understanding of these simple looking objects, in fact already the case of $1/(1-x^4)^{1/2}$ led recently to the ideas that were used to crack Fermat's "last problem" after 350 years.

An example of taking limits of functions would be trying to add up infinitely many functions such as those in the infinite formula for $\sin(x) = x - x^3/3! + x^5/5! - x^7/7!$ ....

"Integrating" a (positive valued) function f means forming an "area" function from f, whose value at x is the area under the graph of f between, (say) 0 and x. This is also a limiting procedure, where we approximate the graph of f by graphs that look like rectangles, then let the area function for f be the limit of the area functions of those rectangular graphs. It turns out that all continuous functions can be integrated and that their area functions (i.e. their integrals) are always differentiable. Moreover if f is continuous, then the derivative of the integral of f, is just f again. Many discontinuous functions can also be integrated, for example all locally bounded monotone functions (such as the greatest integer function **[**x**]**), and the integral of such a discontinuous function, although usually not differentiable, is at least always continuous.

Not all continuous functions can be differentiated (|x| is continuous but not differentiable at 0, and there are continuous functions that are not differentiable anywhere). Although all continuous functions are derivatives (they are the derivatives of their integrals), not all derivatives are continuous; e.g. $(1/x^2).\cos(1/x^2)$ is differentiable everywhere but the derivative is not locally bounded hence not continuous at 0. It is true though that all derivatives do have the intermediate value property that continuous functions have.

You will study integrating functions in math 2310H, and limits of functions such as “Taylor series”, i.e. infinite sums of polynomials, and “Fourier series”, i.e. infinite sums of sine and cosine functions, in math 3100. In fact defining exponential functions and especially proving differentiability for them is so hard, that we usually define the natural log function by an integral, and then define the natural exponential function by inverting the log. This is a big technical advantage because instead of having to prove differentiability for the exponential, which is very hard from scratch [I tried it one night last fall while teaching 2200 and did not succeed, but one of my graduate students showed me how to do it], then we can use the two basic theorems, that all integrals of continuous functions are differentiable and all inverses of differentiable functions with non zero derivative are also differentiable. Exponential and log functions are treated this way in essentially all calculus books, including ours, for this reason. I.e. they all use the process of integrating 1/x to get the natural log function ln(x), and then inverting it to get e^x.

Thus the way the course is actually organized <u>logically</u> is this:
Basic functions:
(i) constants
(ii) x.
Basic operations on functions
(i) algebraic operations,
(ii) composition,
(iii) inversion,
(iv) integration.

Then we generate the trig, exponential, and log functions by inverting the integrals of the reciprocals of x, and the root function $(1-x^2)^{1/2}$. But <u>psychologically</u> we often look at it the first way I described it above. Remember however that I did not even define cos and sin precisely because I assumed we knew how to measure length of arcs on circles. I.e. note that my definition of cos is inverse to the arc length function. If arccos(x) = ø = the radian measure of the angle ø with cos(ø) = x, then ø = the length of the arc of the unit circle from (1,0) to the point $(x, (1-x^2)^{1/2})$. Then cos(ø) is the inverse function of arccos. But what <u>is</u> arccos(x)? I.e. given x how do you find the length of that arc? You can probably find arccos(1), arccos(0), and arccos(1/2), but what is arccos(1/3)? I.e. what is the arclength along the unit circle from (1,0) to $(1/3, (8/9)^{1/2})$? So we may have <u>described</u> this arclength as a least upper bound of lengths of polygons inscribed in the circle, but we do not know how to compute it, or even approximate it well. Indeed we did not even prove the lengths of inscribed polygons have an upper bound, so we do not know they have a least upper bound. The answer to our dilemma is to use integration. I.e. we need integration even to have a satisfactory definition of cos and sin.