**2300H Implicit functions**

This is the study of functions y(x) which satisfy some equation, and how that equation can be used to help compute the derivative of the function. It is also concerned with recognizing which equations are satisfied by some functions. I.e. sometimes we start from an equation and try to use it to define and study a corresponding function.

**Question 1:** Suppose we know that a certain function satisfies an equation, for example suppose we know that the function y(x) satisfies the equation $y^4(x) - y^2(x) + x^2 = 0$, can we use this equation to find the derivative of the function? Well, yes and no. We can just differentiate both sides of the equation using the chain rule, getting $4y^3(x).y'(x) - 2y(x).y'(x) + 2x = 0$. Then since the y' terms are all linear we can easily solve this equation for y', getting $y' = (-2x)/\{4y^3(x) - 2y(x)\}$. This is called "implicit differentiation". I.e. we differentiated an equation satisfied by y, and got an equation for y', and "solved it" for y'. But is this really <u>finding</u> the derivative y'?

We have "solved" for y', but the answer still involves y, so if we did not know a formula for y, then we still do not have a formula for y'. So it is misleading to say that you can find y' even when you cannot find y. Thus I would disagree with the statement that the answer is yes found in some books. Nonetheless this procedure is useful. For example, even if we cannot solve for the function y(x) completely, we might be given the value of the function at a single point. In this case, for instance we might know that y(sqrt(3)/4) = 1/2. Then because we know both x = sqrt(3)/4 and y = 1/2 at this point, we could use the formula for y' above to find the slope of the tangent line to the graph of y(x) at this point.

I.e. plugging (x,y) = (sqrt(3)/4, 1/2) into the formula $y' = (-2x)/\{4y^3(x) - 2y(x)\}$, gives y' = (-sqrt(3)/2) / {4/8 - 1} = sqrt(3). Thus the tangent line to the curve $y^4 - y^2 + x^2 = 0$ at the point (sqrt(3)/4, 1/2) has equation y - 1/2 = sqrt(3)(x- sqrt(3)/4); i.e. y = sqrt(3)x -3/4 + 1/2 = sqrt(3)x -1/4. Do you agree? (or did I slip up?)

One example of a function y(x) that satisfies the equation above is $y_1(x)$ = sqrt{1+ sqrt(1-4$x^2$)}/sqrt(2), which is defined and differentiable for 0 < x < 1/2. There are 3 other such functions on that domain, that satisfy the same equation, namely: $y_2(x) = -y_1(x)$, $y_3(x)$ = sqrt{1- sqrt(1-4$x^2$)}/sqrt(2), and $y_4(x) = -y_3(x)$. Since they all satisfy the same equation, their derivatives all satisfy the same formula. But their derivatives are not the same. How can that be?

Well, the formula for the derivative of $y_1$ has $y_1$ in the formula, while the formula for the derivative of $y_2$ has $y_2$ in its formula, so since these functions are different, their derivatives are different. I.e. above we found the tangent line to the curve at the point (sqrt(3)/4, 1/2), which is on the graph of $y_3$.

I.e. $y_3$(sqrt(3)/4) = 1/2. On the other hand, $y_1$(sqrt(3)/4) = sqrt(3)/2. So the derivative formula $y' = (-2x)/\{4y^3(x) - 2y(x)\}$ at this point gives (-sqrt(3)/2) / (sqrt(3)/2) = -1, I think. Thus the tangent line to the graph of $y_1$ at the point (sqrt(3)/4, sqrt(3)/2) has equation y- sqrt(3)/2 = -(x-sqrt(3)/4) = -x + sqrt(3)/4, or y = -x + (3sqrt(3))/4. Is that right?

**Question 2:** Suppose we are just given the equation, $y^4 - y^2 + x^2 = 0$, and we are not told whether there is any function y(x) satisfying it. Can we decide this just from analyzing the equation? Well, again yes and no. This equation defines a curve in the plane, and we are asking whether some part of that curve looks like a graph of a differentiable function. I.e. whether some part of the curve satisfies the vertical line property (after throwing away some other part), and then whether that part is actually the graph of a differentiable function.

One problem is that there might not be any points on the curve! For example consider $x^4 + y^4 + x^2y^2 + 64 = 0$. The left side is always positive, at least equal to 64, so it can never be 0. So actually there is no function y(x) defined for any x's whatsoever, that satisfies this equation. And it might not always be so obvious whether there are any points on a given equation. But if we can find at least one point satisfied by our equation, then we can tell whether the part of the curve defined by the equation, which is near that point, does or does not look like the graph of a differentiable function.

All you do is take the derivative implicitly and then plug in your point and see whether you get a finite number or not. For example the point (0,1) does satisfy the equation $y^4 - y^2 + x^2 = 0$. Taking the derivative implicitly as before, and pretending that y is a function of x, we get $y' = (-2x)/\{4y^3 - 2y\} = x/(y-2y^3)$. Now we plug in (0,1) and we get $0/(-1) = 0$. This is a perfectly good number, so the "advanced theorem" mentioned in our book, guarantees that the part of the curve defined by the equation $y^4 - y^2 + x^2 = 0$, and lying near the point (0,1), does satisfy the vertical line test, i.e. is the graph of some function, and that function is differentiable.

Moreover, then it follows that the derivative of that function satisfies the equation we just found, i.e. $y' = x/\{y - 2y^3\}$. So basically, given any equation, IF you have a point on it, then go ahead and take the derivative implicitly, and plug in your point. If you get a finite answer, then you have nothing to worry about, i.e. the equation does define a nice curve near your point, the curve does have a non vertical tangent line, and you can find the equation of that tangent line by using your implicit formula. More precisely, there is some small rectangle centered at your point such that the part of the curve which is inside the rectangle, is the graph,of a differentiable function y(x) whose derivative y`(x) satisfies the implicit derivative equation.

But what if you do not get a nice answer when you plug in your point? For example, (1/2, 1/sqrt(2)) satisfies the equation $y^4 - y^2 + x^2 = 0$, but when we plug it into the derivative equation $x/\{y - 2y^3\}$, we get (1/2)/0, not a good answer. Probably none of the book's problems will come out this way, since they will try to avoid such points, but in fact there is a nice tangent line there, but it is vertical. Look at the picture [on page 170, Thomas/Finney, 9th edition], of the figure "eight curve". You could understand that point better by pretending that x was a function of y, and taking the derivative dx/dy implicitly.

Then you would get (remember y is the variable now, and x(y) is the function of y): $4y^3 - 2y + 2x(dx/dy) = 0$, so $dx/dy = (2y-4y^3)/2x = (y-2y^3)/x$. Note this is exactly the reciprocal of what we got before. When we plug (1/2, 1/sqrt(2)) into this we get $0/(1/2) = 0$. So the curve in the picture does pass the horizontal line test near this point, i.e. the curve is the graph of x as a function of y, and the derivative of that function is 0. Either way the equation of the tangent line is $x = 1/2$.

A more interesting (bad) point is (0,0). This also satisfies the equation $y^4 - y^2 + x^2 = 0$, but when we plug it into the derivative formula for y', we get $y' = x/\{y - 2y^3\} = 0/0$, an even worse answer. Looking for x as a function of y will not help this time since the reciprocal of (0/0) will be just as bad. In fact this means that there is no rectangle centered at the point (0,0) in which the curve looks either like the graph of a differentiable function y(x) of x, or of a differentiable function x(y) of y. Looking at the picture of the "eight curve" on page 170 shows what is wrong. No matter how small a rectangle you center at (0,0), the part of the curve inside it still does not pass either the vertical or the horizontal line test. [Near the center point, an "eight" looks like an "x", hence not a graph.]

In this course you will always be given the equation and a point, or if the equation is easy enough you may only be given x and have to solve for y. (That often happens in related rates problems.) You have to take the derivative implicitly, plug in the point and decide whether the curve has a tangent line there or not and write the equation if it does. If you get 0/0 it means either you did the calculation wrong or we messed up and gave you a bad point.

To sum up: if you are given a (polynomial) equation F(x,y) = 0 in x and y, and a point (a,b) satisfying the equation, and you want to know whether there exists a differentiable function y(x) defined on some neighborhood of a, with y(a) = b, and whose derivative can be computed from that equation, just take d/dx of the whole equation and solve for dy/dx in terms of x and y. Then plug in the point (x,y) = (a,b). If you get a finite number c, then the answer is yes, there is such a function y(x), and also y'(a) = c. Moreover if you draw a small enough rectangle in the x,y plane centered at the point (a,b), then the part of the curve defined by the equation F(x,y) = 0 which lies inside the rectangle is the graph of such a function y(x).

On the other hand if you already know there is such a function y(x), and you also know it satisfies the equation F(x,y) = 0, then you may or may not be able to find its derivative from this equation. I.e. if when you solve the equation for dy/dx and plug in (a,b) you get a finite number c, then this number c is the derivative y'(a) of your function. However if you get 0/0 when you plug (a,b) into your formula for dy/dx, but you know y(x) has a derivative at a, it just means the equation you are using is not good enough to compute its derivative. For example, the function y = x has derivative 1 everywhere, and satisfies the equation $y^2 - x^2 = 0$ everywhere. However when I solve this equation for dy/dx, I get 2yy' = 2x, so y' = x/y. Thus at the point (0,0) this does not give me the derivative 1, although it works at all other points.

Worse things can happen, I could have used the equation $(y-x)^2 = 0$. Then I get 2(y-x).y' = 0, so y' = 0/(y-x). Now since our function is y = x, thus y-x is always zero, This gives 0/0 for the derivative at every point. Thus this equation does not work for finding the derivative at any point. So basically the rule is, just solve your equation F(x,y) = 0 for dy/dx and plug in your point. If you get a finite number then everything is fine: i.e. the part of the curve defined by F(x,y) = 0 and lying near your point, is the graph of a function y(x), and the method does give the derivative of that function, hence does give the slope of the tangent line to that curve at that point. If you get a result of form 3/0, or some other non zero number divided by zero, then the curve F(x,y) = 0 has a vertical tangent line at that point. If you get 0/0, the method fails. I.e. either the curve does not have a unique tangent line there or the equation is not good enough to compute it.

**Related rates problems:** These are word problems describing a picture that you have to draw in order to find the equation relating your two functions. Usually the secret to writing an equation is to use one of the basic geometry principles, like Pythagoras' theorem for right triangles, or similar triangles, or some other proportionality principle between corresponding parts of similar figures. Then to complicate things further, the equation will be in terms of x and y, say, but instead of assuming y is a function of x, you will usually assume that both x and y are functions of time t.

Then you differentiate everything with respect to t, and write an equation relating dx/dt and dy/dt. This is why the problems are called related rates problems, since derivatives with respect to time t are called rates of change. Then they will give you enough information, e.g. one of the rates and one of the other measurements, to enable you to solve for the other rate. You will need to use both equations, the one relating the measurements and the one relating the derivatives, that you got by differentiating the first one.

**Example: problem 18, page 177.** Thomas 9th ed. The setup here is an upside down circular cone, and as the water drains out, the water itself forms a shrinking cone similar in shape to the conical reservoir containing it. Now they give the height of the tank as 6m and the radius of the top surface as 45m, and the question they ask is to find rate at which the level of the water is falling, at the instant when the height is 5m, given that the water runs out at a constant rate of 50m^3/min.

Ok, level of water means height of water, and the rate the water runs out is the rate of change of volume of the water. So they are giving us h = 5, and dV/dt = 50, and we want dh/dt.

So first we need an equation involving V and h. If r is the radius of the water at any time, and h its height, then the volume formula for a cone is $V = (\pi/3)(r^2h)$. To get rid of r, we can use the similarity relation between the cone of water and the conical tank. I.e. we get the proportionality equation 6/45 = h/r, or 45h = 6r, hence 15 h = 2r, so r = (15/2)h.

Substituting gives $V = (\pi/3)(r^2h) = (\pi/3)(225/4.h\char`^3) = (75\pi/4)h\char`^3$. This is the equation relating the quantities V and h in the problem. Now we differentiate this equation, i.e. take d/dt of both sides.

That gives dV/dt = (225π/4)h^2.dh/dt. Now we substitute in dV/dt = 50, and h = 5, and we get 50 = (225π/4)(25).dh/dt. Solving for dh/dt, gives dh/dt = [8/(225π)]. Now this is in meters per minute, because the data was given in those units. The gentlemen who wrote the book want the answer in centimeters per minute, but that kind of pointless nonsense annoys me.

Of course there are 100 times more centimeters each minute than meters, so we can just multiply by 100 to get their answer, 800/(225π) = 8\25\4/(9\25π) = 32/(9π) cm/minute.

Now does that seem plausible? I.e. in a minute 50 cubic meters flows out. Since the radius is a bit under 40 meters, the area of that circular water surface is $\pi r^2$ or about 1500π square meters. To lose 50 cubic meters in a minute, the height should change about 1/(30π) ≈ 1/(100) of a meter in that minute shouldn't it? i.e. about 1 cm in a minute, or roughly 32/(9π), so I believe it.