Math 2300H. Bounded and Unbounded Functions
We want to discuss the basic notions of boundedness and unboundedness for functions, which just means whether their values can become arbitraily large in absolute value on their domains or whether there is some absolute value that is never exceeded. Later on in the case of functions that are also differentiable, we will use derivatives to actually find points where some functions take on maximum and minimum values. But not all functions actually assume such maxima or minima, not even all bounded functions. Thus it is useful to know a condition that guarantees a function has a maximum value in a given domain. The most fundamental condition says that if a function is continuous on a closed and bounded interval, then the function is bounded there (above and below) and it takes on a maximum value and also a minimum value.

Notation: "iff" is short for "if and only if".

Definition of boundedness:
Let f be a real valued function defined on a set S. Then we say f is ``bounded on S``, (or simply ``bounded`` if S is understood), if all of the values of f on S are bounded by some fixed number, in absolute value. More precisely, f is ``bounded on S`` if and only if there is some real number M ≥ 0 such that for every x in S, |f(x)| ≤ M.
In symbols, f is bounded on S if and only if (èM≥0)(ìx in S) (|f(x)| ≤ M).
Note this guarantees that f is bounded both above and below, since then, for every x in S, we have -M ≤ f(x) ≤ M.

Remark: If S is a finite set, every function defined on S is bounded on S. Just take the bound M to be the largest of the finite colection of numbers {|f(x)|, for x in S}. Hence boundedness is only in question when the domain S of f is an infinite set, such as an interval of positive length.

Definition of unboundedness: A function f defined on S is unbounded on S, if it is not bounded on S, i.e. if no matter how large we take M to be, there is always some point x in S where |f(x)| > M. In symbols, f is unbounded on S if and only if (ìM≥0)(èx in S) (|f(x)| > M).

Remark: Note how the quantifiers in the unboundedness statement are the opposite of those in the boundedness statement. Moreover it is sufficient to say that |f(x)| can be made larger than any natural number n, i.e. f is unbounded on S if and only if
(ìinteger n ≥ 0)(èx in S) (|f(x)| > n).

Examples The function f(x) = x is continuous and unbounded on the whole real line, but is bounded on every finite interval. The function cos(x) is bounded and continuous on the whole real line and hence also bounded on every finite interval. On the other hand the function g(x) = 1/x is bounded on the infinite interval [1,‘), and unbounded on the finite interval (0,1). It is no accident that g(x) is discontinuous at 0, and has no continuous extension to the closed interval [0,1]. Note also that g(x) is actually unbounded on every interval of form (0,1/n) for every n. The more interesting function h(x) = cos(x)/x is also unbounded on every such interval. (You should graph these functions and be familiar with them.) This phenomenon of being unbounded on every interval about a point, no matter how short the interval, is called ``local`` unboundedness. More precisely,

Definition of local unboundedness: If f is defined on some deleted neighborhood D of a point a (i.e. some set of form (a-∂, a)⁄(a, a+∂)), or on some open interval with a as endpoint (i.e. some set of form (a-∂, a) or (a, a+∂)),but not necessarily at a, then f is ``locally unbounded at a`` (or ``locally

unbounded near a``) if and only if f is unbounded on <u>every</u> deleted neighborhood of a. In symbols, f is ``locally unbounded at a`` if and only if no matter how large we take M to be, and no matter how small we take $\partial$ to be, there is some point x in $(a-\partial, a+\partial)$ where f has a value larger than M in absolute value. In symbols, f is ``locally unbounded at a`` if and only if
(ìM ≥ 0)(ì$\partial > 0$)(èx)such that ($0 < |x-a| < \partial$ and $|f(x)| > M$).

Definition of local boundedness: If f is defined on some deleted neighborhood D of a point a, or on some open interval with a as endpoint, then f is ``locally bounded at a`` (or ``locally bounded near a``) if and only if f is bounded on <u>some</u> deleted neighborhood of a. Equivalently, there is some bound M and some deleted ∂ neighborhood of a where f is bounded by M, in absolute value. In symbols, f is ``locally bounded at a`` if and only if
(èM ≥ 0)(è$\partial > 0$)(ìx)($0 < |x-a| < \partial$ implies $|f(x)| \leq M$).
In case f is only defined say to the right of a, this would read
(èM ≥ 0)(è$\partial > 0$)(ìx)($0 < x-a < \partial$ implies $|f(x)| \leq M$).
Continuous functions are locally bounded. This is a basic property of continuous functions. More precisely,
Theorem: If f is continuous at a, then f is locally bounded at a.
proof: Take any positive number Ÿ > 0, (such as Ÿ = 1). then by definition of continuity there is some $\partial > 0$ such that for all x with
$|x-a| < \partial$, we have $|f(x) - f(a)| <$ Ÿ. We claim then M = (|f(a)| + Ÿ) is a bound for f on the interval $(a-\partial, a+\partial)$. I.e. for all x with $|x-a| < \partial$, we have $|f(x) - f(a)| <$ Ÿ, but in general $|A| - |B| \leq |A-B|$, so for $|x-a| < \partial$, we have $|f(x)| - |f(a)| \leq |f(x) - f(a)| <$ Ÿ, thus $|f(x)| - |f(a)| <$ Ÿ. Hence by adding |f(a)| to both sides of this inequality, we get $|f(x)| <$ |f(a)| + Ÿ = M, for all x with $|x-a| < \partial$. This proves f is locally bounded near a. **QED.**

Note this means that a function like f(x) = 1/x can be <u>locally</u> bounded at every point of (0,1), since it is continuous at every point of (0,1), and yet be <u>unbounded</u> <u>globally</u> on (0,1). This is only possible because the interval (0,1) is open. I.e. we have the following basic theorem:

Theorem: If f is locally bounded at every point of the closed bounded interval [a,b], then f is globally bounded on [a,b].
proof: We know f is locally bounded near a, so at least there is some x with $a < x < b$ such that f is bounded on the interval [a,x]. We just want to show we can move x to the right all the way to b and still have f bounded. So consider the set S = { those points x in [a,b] such that f is bounded on [a,x] }. Then a belongs to S, but all elements of S are bounded above by b, so S has a least upper bound, say L. Then we can conclude that f is bounded on [a,x] for every x with $a \leq x < L$, but f is not bounded on [a,x] for any x with $L < x \leq b$.
We claim L = b. We will prove it by contradiction. Since a is in S and L is an upper bound for S, we see $a \leq L$. Since b is an upper bound for S and L is the least upper bound, we see also that $L \leq b$, so $a \leq L \leq b$. We claim L cannot be less than b. For if $L < b$, then since f is continuous at L, by assumption, f is locally bounded at L so there is some $\partial > 0$ such that f is bounded by some M_1 on $(L-\partial, L+\partial)$ and by taking ∂ smaller if necessary, we will have $L < L+\partial < b$. Hence f is bounded by M_1 on the interval $[L- \partial/2, L+ \partial/2]$ and, since $L - \partial/2$ is less than L, also f is bounded on the interval $[a, L- \partial/2]$ say by M_2. Then if $M = \max\{M_1, M_2\}$, f is bounded by M on the whole interval
$[a, L+ \partial/2]$. But this contradicts the fact that f is not bounded on [a,x] for any x with $L < x \leq b$. I.e. $L + \partial/2$ would be such an x ~ This contradiction shows that in fact $L < b$ is impossible, so L = b.
Now we know that f is bounded on all intervals of form [a,x] with
$x < b$, and we claim that in fact f is bounded on [a,b]. Also f is locally bounded at b, say by N_1, on

some ∂ - neighborhood of b, i.e. on (b-∂,b) for some ∂ > 0. On the other hand, since b - ∂/2 < b = L, f is also bounded on [a, b - ∂/2] say by N_2. Then f is bounded on [a,b) by max{N_1, N_2}. We have to include the endpoint b also but that is easy since it is only one point. I.e. if we take N = max{N_1, N_2, |f(b)|}, then f is bounded by N on [a,b]. **QED.**

Corollary: In particular, since every continuous function on [a,b] is locally bounded at every point, every continuous function f on [a,b] is globally bounded on [a,b].

Now from the corollary we know the set of values of a continuous function f on [a,b] have an upper and a lower bound, and hence by the least upper bound axiom for real numbers, there is a least upper bound and a greatest lower bound for those values. We claim that for a continuous f on [a,b] that the lub of its values is actually a value, the ``maximum value``, and the glb of the values is also a value, the ``minimum value``. Thus a continuous function on a closed bounded interval assumes a maximum value and a minimum value on [a,b].

More precisely,
Theorem: If f is continuous on [a,b] and if M is the lub of its values on [a,b] while m is the glb of those values, then there is some point x_0 in [a,b] where f(x_0) = m, and some point x_1 in [a,b] where f(x_1) = M.
proof: (for M). Suppose to the contrary that f never takes the value M. Since M is the least number ≥ all the values of f on [a,b], then f takes values arbitrarily near M from below, i.e. for every n > 0, there is a value of f between M - 1/n and M. Thus for every n > 0, there is some x in [a,b] such that M - 1/n < f(x) < M. Thus the reciprocal function 1/(M-f(x)) is greater than 1/(1/n) = n at x. Thus if f never equals M, then the reciprocal function 1/(M-f) is both continuous and unbounded on [a,b]. This is a contradiction to the previous corollary. **QED.**

Corollary: If f is continuous on [a,b] then there are points x_0 and x_1 in [a,b] such that for every x in [a,b], we have f(x_0) ≤ f(x) ≤ f(x_1). We say f takes its maximum value at x_1 and its minimum value at x_0.

Definition of infinite limits
A function f defined in a deleted neighborhood of a is said to have limit +‘ at a if no matter how large you ask the values to be, on some deleted ∂ neighborhood of a all values of f(x) are that large. I.e.
$\lim_{x \to a}$ f(x) = +‘ if and only if for every M ≥ 0, there is some deleted ∂ neighborhood of a on which all values of f are at least M. In symbols, $\lim_{x \to a}$ f(x) = +‘ if and only if
(ìM ≥ 0)(è∂ > 0): (ìx) (0 < |x-a| < ∂ implies f(x) ≥ M).

Similarly, $\lim_{x \to a}$ f(x) = -‘ if and only if
(ìM ≥ 0)(è∂ > 0): (ìx) (0 < |x-a| < ∂ implies f(x) ≤ - M).

One sided infinite limits: f has limit +‘ as x approaches a from the right if no matter how large M is, all values of f at points x near enough to a and on the right side of a are at least M. In symbols, $\lim_{x \to a^+}$ f(x) = +‘ if and only if
(ìM ≥ 0)(è∂ > 0): (ìx) (0 < x-a < ∂ implies f(x) ≥ M).

Negative infinite limits from the right, and infinite limits from the left, are defined similarly and

denoted $\lim_{x \to a^+}$ f(x) = -‘, or
$\lim_{x \to a^-}$ f(x) = +‘, or $\lim_{x \to a^-}$ f(x) = -‘.

We can also define ``limits at infinity``, which means as x approaches either plus infinity or minus infinity, as follows. The point here is that being near infinity means being larger than some large number. Hence $\lim_{x \to +‘}$ f(x) = L, where L is a finite number, if and only if for every Ÿ > 0, there is some neighborhood of +‘ on which all values of f are in the Ÿ neighborhood of L.
I.e. (ìŸ > 0)(èM ≥ 0): (ìx) (x ≥ M implies |f(x) - L| < Ÿ).
Similarly, $\lim_{x \to -‘}$ f(x) = L if and only if
(ìŸ > 0)(èM ≥ 0): (ìx) (x ≤ - M implies |f(x) - L| < Ÿ).

Of course f can also have an infinite limit at ‘,
$\lim_{x \to +‘}$ f(x) = +‘ if and only if no matter how large you want f(x) to be, all values f(x) will be that large, if x is large enough, i.e.
(ìM ≥ 0)(èN ≥ 0): (ìx) (x ≥ N implies f(x) ≥ M).

How to recognize unbounded functions and infinite limits.
Proposition: (i) If f = g/h and if h approaches 0 at a but g does not approach 0 at a, then f is locally unbounded at a, (whether g has a limit at a or not).
(ii) If g has a positive limit at a and h approaches zero through positive values, then $\lim_{x \to a}$ f(x) = +‘.
(iii) If g is locally bounded at a (for example if g is continuous at a), and h has limit ‘ (or -‘) at a, then f = g/h has limit zero at a.

Other combinations in (ii) are possible with other signs.

Example: A rational function f = g/h where g,h are polynomials. If a is a zero of both the numerator and denominator, then one can cancel factors of (x-a) until this is no longer true. Then f is locally unbounded at every zero of the denominator h which is not a zero of the numerator g. At a simple zero of h which is not a zero of g, g/h is positive on one side of a and negative on the other side. Then f has limit infinity from the side on which g/h is positive, and limit minus infinity from the other side; in particular g/h does not have a (two sided) limit at a, but |g/h| does have (two sided) limit = +‘ .

Cases where the limit looks like ``0/0``, or ``‘/‘``. If both g and h have limit zero at a, or if both have limit ‘, then we must test further to determine whether f has a limit at a or not. If g , h are polynomials and a is a root of both, then we can cancel a factor of
(x-a) from both g and h and retest. The best tool for more general differentiable functions is ``l`Hopital`s rule``, (due to Bernoulli). It says if g and h are both differentiable on some deleted neighborhood of a, and if both g and h have limit 0 at a, or if both have infinite limits at a, then we may replace g/h by the quotient g`/h` of their derivatives and test it instead. I.e under these assumptions, if g`/h` has either a finite or an infinite limit at a, then g/h has that same limit. Moreover, the same thing holds for one sided limits at a, and for limits as x approaches either ‘ or -‘.

Do the easy cases of the chain rule such as f^n, $f^{1/q}$, $f^{p/q}$.