Intermediate Values
An intuitive property we expect for continuous functions is that the graph should not have "holes" or "gaps" in it, whatever that means. One way to make this precise is to say that if the graph starts out on one side of a horizontal line and ends up on the other side, then the graph meets that line in between. Of course this is not always true. For instance if f is defined on the domain made up of two separate intervals, such as [0,1] and [2,3], and say $f(x) = x$ for all x in Domain(f), where Dom(f) = [0,1] / [2,3], then f is continuous at every point of its domain, since f is a polynomial, but the graph starts out below the line $y = 3/2$ (i.e. $f(1) = 1 < 3/2$) and ends up above it, since $f(2) = 2 > 3/2$, but the graph never crosses the line $y=3/2$, since it never meets it. What gives?

The problem of course is that the domain of f was not one interval but two disjoint ones. So the first requirement is that the domain of f should be an interval. Even then, if Dom(f) = [1,3], and if $f(x) =$ **[x]** = the "greatest integer function", i.e. $f(x)$ = the largest integer not larger than x, then the graph of f starts out below the line $y = 7/5$, since $f(1) = 1 < 7/5$, and ends up above that line since $f(3) = 3 > 7/5$, and yet $f(x)$ never equals 7/5 anywhere between 1 and 3, so the graph never crosses that line. So the second property that can make the theorem fail is lack of continuity.

However if those properties are both present then the intermediate value property is true:
IVT (intermediate value theorem):
If f is defined and continuous at every point of an interval I, (either open or closed or half open, finite or infinite or half infinite), and if a and b are two, points of I, and if M is any number such that either $f(a) < M < f(b)$, or $f(b) < M < f(a)$, then there is at least one number c in the open interval (a,b) such that $f(c) = M$.
proof: (We are going to prove this theorem using least upper bounds.)
Assume for simplicity that $f(a) < M < f(b)$. (The other case is proved in a very similar way.) The idea is that if we look at all numbers x in [a,b] such that $f(x) < M$. Then at the very "next" number, f should have value = M. To make this idea of a "next number" precise, we use the notion of "least upper bound".

Let S = {those points x such that $a \le x \le b$ and $f(x) < M$}. Then S is not the empty set since a is in it, i.e. by hypothesis we have $f(a) < M$, so a is in the set S. Moreover the numbers in S are by definition no larger than b, so b is an upper bound for S. Then by the lub axiom, since S is non empty and bounded above, it has a <u>least</u> upper bound, i.e. there is a smallest number c which is not smaller than any number in S.

OK? If x is in S, i.e. if $f(x) < M$, then $x \le c$, but no smaller number has this property. So if $\partial < c$ then some x in S is larger than ∂. I.e. if $\partial < c$, then there is some x with $\partial < x < c$ and $f(x) < M$.

CLAIM: (i) $a \le c \le b$, and (ii) $f(c) = M$.
proof of i) Since b is an upper bound and c is the least upper bound, we must have $c \le b$.

(ii) This part we have to prove by contradiction. So we have to show it would not be possible either to have $f(c) < M$ or $f(c) > M$. So first assume that $f(c) < M$. Then we claim there are some numbers in S that are greater than c, contradicting the fact that c is an upper bound for S.

I.e. we claim that if $f(c) < M$, then there are some numbers x in [a,b] that are greater than c and that satisfy $f(x) < M$. First of all we need to show there are some numbers greater than c in [a,b], i.e. we need to show that $c < b$. We know $a \le c \le b$, and $f(b) > M$, while $f(c) < M$, so c cannot equal b, hence $c < b$. So there is some room to the right of c in the interval [a,b]. Now we know $c < b$, so take ∂_1 so small that $c+\partial_1 < b$ also; say let $\partial_1 = (b-c)/2$.

Next we use continuity of f at the point c. Since $f(c) < M$, then for any point x very near and to the right of c, we will have f(x) very near f(c). But since $f(c) < M$, if we make f(x) close enough to f(c), then f(x) should also be less than M. I.e. take $e = (M-f(c))/2$. Then there is a $\partial_2 > 0$ such that whenever $L-\partial_2 < x < L+\partial_2$, then f(x) is closer to f(c) than $(M-f(c))/2$. Now let $\partial = \min\{\partial_1, \partial_2\}$.

If we take any x such that $c < x < c+\partial$, say $x = c + \partial/2$, then we have $c < x < c+\partial < b$, and also $f(c) - e < f(x) < f(c) + e$. We claim this implies $f(x) < M$.

At least $f(x) < f(c) + e$, i.e. $f(x) < f(c) + (M-f(c))/2$. If we simplify the right side we get $f(x) < f(c) + (M-f(c))/2 = f(c) + M/2 - (f(c))/2 = \{f(c) + M\}/2 < (M + M)/2 = M$.
(We used the fact that $f(c) < M$).

Thus tracing through the inequalities, we get $f(x) < M$. Since also $a \leq c < x < c+\partial < b$, this x is in [a,b] hence in S, but x is larger than c, a contradiction.

The argument is picky, but the idea is very simple: c is supposed to be to the right of all the numbers x with $f(x) < M$, but if $f(c) < M$, then any x close to c and to the right (or left) of it, would also have $f(x) < M$, a contradiction.
Challenge: Prove also that $f(c) > M$ gives a contradiction.
(Remark: I like my more elementary proof with infinite decimals better.)