2300H
Recall our fundamental theorems about continuity.

1) Intermediate value theorem. (IVT)
Short version: If f is defined and continuous on an interval I, then the set of values f takes on I, is also an interval.

Explicit version: If f is defined and continuous on an interval I, if a,b are points of I with a < b, and if M is a number such that either f(a) < M < f(b) or f(b) < M < f(a), then there is a number c in I such that a < c < b and f(c) = M.

(The two previous statements are equivalent.)

2) Maximum and minimum value theorem. (MMVT)
Short version: If f is defined and continuous on a closed bounded interval I, then the set of values f takes on I, is also a closed bounded interval.

Explicit version: If f is defined and continuous on a closed bounded interval I = [a,b], then there are points c, d in I such that for every x in I, f(c) ≤ f(x) ≤ f(d). We say f(c) is the minimum value of f on I and f(d) is the maximum value of f on I.

(This time the explicit version is weaker than the short version. The short version is equivalent to the explicit version plus the IVT.)

Construct examples of functions defined and continuous on bounded open intervals, where the set of values is bounded and closed, or bounded and open, or half open and bounded, half open and unbounded, half closed and bounded, half closed and unbounded, or all of R. The same can be done for functions defined and continuous on half open bounded intervals. Construct examples of continuous functions on half closed unbounded intervals where the set of values is bounded and closed, or bounded and open, or half open and bounded, half open and unbounded, half closed and bounded, or all of R.

l'Hopital`s rule: If f and g are both differentiable and both approach 0, or both have infinite limits, as x approaches a, then we can consider the limit of f'/g' instead. If this quotient has a limit as x approaches a, either a finite or infinite limit, then f/g has the same limit.

BUT DO NOT try to use this rule when the underlined hypotheses on f and g do not hold!

What the derivative tells you about the behavior of a function.

We all know that the derivative is supposed to tell us whether the function is increasing or decreasing or has a "max" or "min", but exactly how does it do this? We will discuss these points here, with proofs, which are relatively easy, at least compared with the hard proofs of the theorems above.

Verbum sapienti: (words to the wise):
Draw pictures for yourself to illustrate all the ideas and results below.

First let us make precise the two notions of "increasing" that we will use.
Definition:
1) A function f is strictly increasing <u>on the interval I</u>, if and only if, f is defined on I and for all x,y in I, if $x < y$, then $f(x) < f(y)$.
2) A function f is strictly increasing <u>at the point c</u>, if and only if, there is some $e > 0$, such that for all x, if x is in Dom(f) and $c-e < x < c$, then $f(x) < f(c)$, and if x is in Dom(f) and $c < x < c+e$, then $f(x) > f(c)$.

Remark: 1) It is possible for f to be strictly increasing at an endpoint of its domain. For instance to be strictly increasing at the right endpoint c of Dom(f) means there is some interval to the left of c on which f always has smaller values than at c.
2) Notice that the definition allows f to be increasing "at the point c" but not increasing on any interval containing c. The reason is that in the definition of increasing at c, we require one of the two points always to be c, while for increasing on an interval we use any two points in the interval. Actual examples of this can be given and we "drew" one in class.

Theorem 1: If f is defined on some interval containing c, and if $f'(c)$ exists and is positive, then f is strictly increasing at c.
Proof: The only property of limits we will use is that if a function g has a positive limit at c, then there is a punctured open interval containing c where the values of the function are all positive. Now $f'(c) > 0$ means the function $g(x) = \{f(x)-f(c)\}/(x-c)$, has positive limit at c. Thus there is some $e > 0$ such that if $c-e < x < c$, then $g(x) > 0$, and also if $c < x < c+e$, then $g(x) > 0$.

I.e., if $c-e < x < c$, then $\{f(x)-f(c)\}/(x-c) > 0$, and since the denominator is negative, then the numerator must be negative, i.e. $f(x) - f(c) < 0$, so $f(x) < f(c)$.

On the other hand, if $c < x < c+e$, then again $g(x) = \{f(x)-f(c)\}/(x-c) > 0$, but now the denominator is positive, so the numerator is positive, i.e. $f(x) > f(c)$. **QED.**

Remark: We define strictly *decreasing* <u>on an interval,</u> and strictly *decreasing* <u>at a point</u> c, in analogous ways, and then one can prove in the same way, that if f has negative derivative at a point c of its domain, then f is strictly decreasing at c.

Definition:
1) f is <u>strictly</u> <u>monotone</u> <u>on the interval I</u> if and only if f is either strictly increasing or strictly decreasing on I.
2) f is <u>strictly monotone at c</u> if and only if f is either strictly increasing at c, or strictly decreasing at c.

Definition:

(i) f has a <u>local</u> <u>minimum</u> (or relative minimum) at c in Dom(f) if and only if there is some $e > 0$ such that for all x in Dom(f), if $c-e < x < c+e$, then $f(x) \geq f(c)$.
(ii) A <u>local</u> <u>maximum</u> (or relative maximum) for f is a point c in Dom(f) such that there is some $e > 0$, such that for all x in Dom(f), if $c-e < x < c+e$, then $f(x) \leq f(c)$.
(ii) A <u>local</u> <u>extremum</u> (or relative extremum) for f is a point c in Dom(f) which is either a local minimum or a local maximum for f.

Remark: Note that if f is strictly monotone at an endpoint c of Dom(f), then c is also a local extremum of f. In particular it is possible for a local extremum of f to be at an endpoint of Dom(f).

Corollary 2: If c is an endpoint of Dom(f) where $f'(c) \neq 0$, then c is a local extremum of f.
Proof: Suppose c is the left endpoint of Dom(f) and $f'(c) < 0$. Then f is strictly decreasing at c. Since f is only defined to the right of c and not to the left, then on some neighborhood of c, all values f has are smaller than f(c). I.e. c is a local maximum. The other 3 cases are similar. **QED.**

Definition: A <u>critical</u> <u>point</u> of a function f is a point c where f(c) is defined, but either f'(c) is not defined, or $f'(c) = 0$.

Definition: A point c is an interior point of Dom(f) if and only if for some $e > 0$, the interval (c-e, c+e) is contained in Dom(f), i.e. not only is c contained in Dom(f) but some interval centered at c is contained in Dom(f).

Fundamental fact:
Theorem 1 above implies that local extrema can <u>only</u> occur at endpoints or critical points.

Corollary 3: If f has a local extremum at a point c of Dom(f), then at least one of the following is true:
(i) c is an endpoint of Dom(f), or
(ii) f' is not defined at c, or
(iii) $f'(c) = 0$.
Proof: To show at least one of those must be true at a local extremum, it suffices to assume that if two of them fail, the third must hold, i.e. that they cannot all fail. So assume that f has a local extremum at c in Dom(f), and that c is not an endpoint of Dom(f) but that f'(c) is defined. then we must show $f'(c) = 0$. We do this by showing that f'(c) cannot be either positive or negative. For if it were positive then by the theorem above, f would be strictly increasing at c, and since c is not an endpoint of Dom(f), there would be points to the left of c where f has smaller value and points to the right of c where f has larger value. this would contradict c being a local extremum. Similar reasoning shows f'(c) cannot be negative, so we must have $f'(c) = 0$. **QED.**

Remarks:
i) The conclusion is that if you are looking for local extrema, you only have to look at end points and critical points; i.e. if there are no local extrema at those points, then there are no local extrema.
ii) The converse of the Corollary is not true; endpoints and critical points do not have to be local extrema.
I.e. many functions do not have local extrema at critical points, such as $f(x) = x^3$, $f(x) = x^{1/3}$, $f(x) = x^5$, $f(x) = x^{1/5}$,......, which all have critical points at $x = 0$. Other functions can have several critical points,

some of which are local extrema and some of which are not, such as $f(x) = 3x^4 + 4x^3$, which has critical points at $x = \{-1, 0\}$, with a local min at -1, but is strictly increasing at the critical point $x = 0$.

In particular the result proved above about strict monotonicity at a point cannot be reversed either, i.e. a function can be strictly increasing at a point without the derivative being positive there.

iii) Always be careful of assuming anything about the behavior of a function at a point where the derivative is zero. The function can do almost anything at such a point, increase, decrease, have a local max, a local min, or none of the above.

For example, if $f(x) = x^2 \cos(x)$ for $x \neq 0$, and $f(0) = 0$, then $f'(0) = 0$ but f is neither increasing nor decreasing at 0 and has no local extremum there either, but oscillates up and down infinitely many times as x approaches 0.

I repeat:
iv) Do not assume anything about a function at a point c where $f'(c) = 0$. Almost all students get this exactly backwards, by thinking that if $f'(c) = 0$ then the point must be a local max or local min.
This is wrong!
Knowing that $f'(c) = 0$ is NEVER enough to conclude <u>anything</u> about the behavior of f at c. You must ALWAYS perform some further tests on this point. All $f'(c) = 0$ tells you is that this is POSSIBLY a local max or min.
I.e. the theorem is a negative one; it says if $f'(c)$ is NOT zero, and if c is an interior point of Dom(f), then c is NOT a local extremum. Notice however that if c is an endpoint of Dom(f) and if $f'(c)$ is NOT zero, then c is a local extremum of f. (Draw a picture!) In general you CAN deduce something about the behavior of f near c just from knowing $f'(c) \neq 0$, but not just from knowing $f'(c) = 0$.

To have sufficient criteria for interior local extrema, we derive an important corollary of MMVT above on the existence of "global" extrema.

Definition:
(i) If c is a point of Dom(f), we say f has a "maximum" or "absolute maximum", or "global maximum" at c, if and only if for all x in Dom(f), we have $f(x) \leq f(c)$.
(ii) If c is a point of Dom(f), we say f has a "minimum" or "absolute minimum", or "global minimum" at c, if and only if for all x in Dom(f), we have $f(x) \geq f(c)$.
(iii) A point c in Dom(f) is a global (or absolute) extremum for f if and only if c is either a global max or a global min.

Remark: The difference between global/absolute extrema and local/relative extrema, is that in the local case we do not have to name the interval on which c is the extremum. I.e. c is a local extremum if it is an extremum on some unspecified e - interval around c, but to be a global extremum it has to be the extremum on the whole specified domain. So c is a local extremum for f if and only if there is some $e > 0$ such that c is a global extremum on the new smaller domain (c-e, c+e) intersected with Dom(f).

Maybe it would be better to define it this way:
Definition:

(i) f has a maximum at c on the set S, if and only if f is defined on S, and S contains c, and $f(x) \leq f(c)$ for all x in S.
(ii) We call c a "global maximum" for f if f has a maximum at c on the set Dom(f).
(iii) We call c a local maximum for f if there is some $e > 0$ such that f has a maximum at c on the set (c-e,c+e) intersected with Dom(f).

Remark: We define minima, global minima, and local minima, global extrema, and local extrema similarly.

Lemma 4: A global extremum is also a local extremum, (a global max is also a local max, and a global min is also a local min).
Proof: If $f(x) \leq f(c)$ for all x in Dom(f), then for any $e > 0$, we also have that $f(x) \leq f(c)$ for all x in (c-e, c+e) intersected with Dom(f). (The minimum case is similar.) **QED**

The next result uses our heavy duty boundedness theorems, not just definitions.

Rolle's Theorem: If f is continuous on the closed bounded interval [a,b] and differentiable at least on (a,b), and if f(a) = f(b), then there is at least one point c such that:
(i) $a < c < b$, and
(ii) $f'(c) = 0$.
Proof: We will show that under these assumptions at least one of the global extrema guaranteed by the MMVT occurs at an interior point c, and that will give us an interior local extremum, so f'(c) will be 0.
Case 1) f is constant on [a,b]. If $c = (a+b)/2$, then $a < c < b$, and since f is constant, $f'(x) = 0$ for all x in [a,b], so $f'(c) = 0$.
Case 2) f is not constant on [a,b]. Then f has a (global) max and a (global) min on [a,b] and since f(a) = f(b), the value f(a) = f(b) cannot be both the max and the min. Hence one of the global extrema occurs at an interior point c. Then c is an interior local extremum, so by corollary 3, $f'(c) = 0$. **QED.**

Rolle's theorem combined with the IVT gives a very important corollary. This is the basic princiople of graphing, but I cannot think of a single book in which this principle is explained. Of course once I say that, many of them will leap forward. Be my guest. (The result is easy for functions whose derivative is continuous, but the point here is that continuity of the derivative is unnecessary. I.e. all derivatives f' have the intermediate value property, even those f' that are not continuous.)

Corollary 5: If f is continuous on an interval I and f has no critical points in the interior of I, then f is strictly monotone on I.
Proof: (We prove it by contradiction; briefly, if f is <u>not</u> strictly monotone on I, then by the IVT, f takes some value twice, and then Rolle forces a critical point.)

First, since f has no critical points in the interior of I, then f is differentiable everywhere in the interior of I, so by Rolle, f cannot ever take the same value twice at any two points of I. But if f is not strictly monotone then f is neither strictly increasing nor strictly decreasing on I. Thus there exist points a,b,c,d in I, with $a < b$ and $f(a) < f(b)$, and $c < d$, and $f(c) > f(d)$. We do not know exactly how the points a,b,c,d are ordered, nor whether they are all different, but we know f fails to be strictly monotone on this set of points. However since f is strictly monotone on any set of two points, at least three of them must be different.

We want to show that f fails to be strictly monotone on some subset of three of these points. (There are several cases, but the arguments are similar so we shall give only a representative sample of the full argument.) If only three of the points are distinct the assertion follows immediately. So assume all four points are distinct.

If the points are ordered as $a < b < c < d$, then either $f(c) \geq f(b)$ or $f(c) < f(b)$. If $f(c) \geq f(b)$, then f is not strictly monotone on the three points a,c,d. If $f(c) < f(b)$, then f is not monotone on the three points a,b,c.

If we have instead $a<c<b<d$, and $f(c) \geq f(b)$, then f is not monotone on the three points a,c,d. If $f(c) < f(b)$, the f is not monotone on the three points a,b,c.

The other cases are exactly similar. **QED.**

Remark: Corollary 5 tells us most of what we need to know to graph functions, i.e. just plot the critical points and then "connect the dots". All we have to know besides this, is what the function does past the last critical points, i.e. what the limits at infinity are, and where it changes concavity, i.e. where the "inflection points" are.

Next we have several ways to recognize local and global extrema.

Tests for local extrema
Theorem: : Assume that f is continuous on Dom(f), and that Dom(f) is an interval.
(i) endpoint test: If c is an endpoint of Dom(f) and $f'(c) \neq 0$ then c is a local extremum of f. More precisely, a left endpont c is a local max if $f'(c) < 0$, and a local min if $f'(c) > 0$. A right endpoint behaves opposite to this.
(ii) zeroth derivative test: If c is an interior point of Dom(f), and if there are points a, b in Dom(f) such that f is continuous on [a,b] and c is the only critical point of f in (a,b), then c is a local min of f if and only if f(a) and f(b) are both larger than f(c), and c is a local max if and only if f(a) and f(b) are both smaller than f(c).
(iii) first derivative test: If c is an interior point of Dom(f), and if there are points a,b in Dom(f) such that $a < c < b$, f is continuous on [a,b], and c is the only critical point of f in [a,b], then c is a local min of f if and only if $f'(a) < 0$ and $f'(b) > 0$, and c is a local max if and only if $f'(a) > 0$ and $f'(b) < 0$.
(iv) second derivative test: If c is an interior point of Dom(f) at which $f'(c) = 0$ and $f''(c) > 0$, then f is concave up at c, so c is a local min of f. If $f'(c) = 0$ and $f''(c) < 0$, the f is concave down at c, so c is a local max.

Tests for global extrema:
One critical point case: If f is continuous on the (possibly infinite) open interval (a,b) and if c is the only critical point of f in (a,b) then c is a global min if and only if it is a local min, and a global max if and only if it is a local max.

If f has only one critical point, then (since the interval is open) f will have at most one global extremum, either a max or min, but not both, and it may not have either one.

Assume f continuous with only one critical point on (a,b) at c. Then we have:
(i) zeroth derivative test:
If there are points x_1, x_2 with $a < x_1 < c < x_2 < b$, such that $f(x_1) < f(c) > f(x_2)$, then c is a global max for f on (a,b).

If there are x_1, x_2 with å < x_1 < c < x_2 < ∫, such that $f(x_1) > f(c) < f(x_2)$, then c is a global min for f on (a,b).

(ii) first derivative test:
If there are points x_1, x_2 with $a < x_1 < c < x_2 < b$, such that $f'(x_1) < 0$, and $f'(x_2) > 0$, then c is a global min for f on (a,b).

If there are x_1, x_2 with $a < x_1 < c < x_2 < b$, such that $f'(x_1) > 0$, and $f'(x_2) < 0$, then c is a global max for f on (a,b).

(iii) second derivative test:
If $f''(c) < 0$, then c is a global max for f on (a,b).

If $f''(c) > 0$, c is a global min.

(iv) limit test:
If f(x) --> $+\infty$, both when x -->a^+ and when x-->b^-, then c is a global min.

If f(x) --> $-\infty$, both when x -->å$^+$ and when x-->b^-, then c is a global max.

The previous tests are sufficient for most applied max/min problems. The following generalizations are useful in graphing problems where there are more critical points. You must draw pictures to see these conditions easily. (Memorizing all these inequalities is hopeless.) These more complicated cases essentially never come up in book problems, but I could not resist stating them.

More generally:
Assume f is continuous on Dom(f), and f has a finite number of critical points: $\{c_1,....,c_n\}$.
Closed bounded interval test: If Dom(f) is a closed bounded interval [a,b], then f always has both a global min and a global max, and the smallest of the values f(a), $f(c_1)$,....,$f(c_n)$, f(b), is the global min of f on [a,b] and the largest of these values is the global max.

Open interval tests:

If Dom(f) is an open interval (a,b), choose *any* two points x_1, x_2 with $a < x_1 < c_1$ and $c_n < x_2 < b$.

(i) zeroth derivative test: If $f(x_1) < f(c_1)$ and $f(x_2) < f(c_n)$, then the largest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global max of f on (a,b), (but f may not have a global min).

If $f(x_1) > f(c_1)$ and $f(x_2) > f(c_n)$, then the smallest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global min of f on (a,b), (but f may not have a global max).

(ii) first derivative test: If $f'(x_1) > 0$ and $f'(x_2) < 0$, then the largest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global max of f on (a,b), (but f may not have a global min).

If $f'(x_1) < 0$ and $f'(x_2) > 0$, then the smallest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global min of f on (a,b), (but f may not have a global max).

(iii) limit test: If f(x) --> $-\infty$, both when x -->a^+ and when x-->b^-, then the largest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global max of f on (a,b), (but f may not have a global min).

If f(x) --> $+\infty$, both when x -->a^+ and when x-->b^-, then the smallest of the values $f(c_1)$, $f(c_2)$,.....,$f(c_n)$ is the global min of f on (a,b), (but f may not have a global max).

As an interesting corollary we can prove, as mentioned above, that all derivatives have the intermediate value property. I.e. If f is differentiable, then of course f is continuous and thus has the intermediate value property. But in fact the derivative f' also has the intermediate value property, whether f' is continuous or not. This is the basic fact that is omitted in most books, (except for Courant).

Corollary 6: Every derivative f', has the Intermediate value property.
Proof: Suppose f is defined and differentiable on the interval I and there are points a < b in I such that f'(a) < 0 while f'(b) > 0. We claim there is a point c with a < c < b such that f'(c) = 0. We know f is strictly increasing at b and strictly decreasing at a, so f is not monotone on [a,b], and hence not monotone on (a,b). Thus f has critical ponts in (a,b). But f is differentiable on (a,b), so f' equals zero somewhere on (a,b).

Now suppose f'(a) = A and f'(b) = B and C is a number with A < C < B. If we define g(x) = f(x) - Cx, then g is differentiable and g'(a) = A - C < 0, while g'(b) = B - C > 0. Hence by what we have proved in the previous paragraph, there is some c with a < c < b, such that g'(c) = 0. Since g'(c) = f'(c) - C, thus f'(c) = C.
QED.

We can also derive the usual criterion for increasing and decreasing functions.
Corollary 7: If f'(x) > 0 for all x in (a,b), then f is strictly increasing on (a,b).
Proof: We know that f is strictly monotone on (a,b) because it has no critical points, so it is either strictly increasing or strictly decreasing on (a,b). Since it is also strictly increasing at every point since f' > 0, it must be strictly increasing on (a,b). **QED.**

We can also prove that if $f' < 0$ on (a,b) then f is strictly decreasing the same way.

Remark: These are the criteria in most books, but note using them requires checking the value of the derivative at every point of the domain. Our earlier criteria are much easier to use: if there are no critical points in the interval Dom(f) then f is strictly increasing if there is even one point c where $f'(c) > 0$.

Another important corollary of Rolle's theorem we will use later is:
Mean Value theorem:
If f is continuous on $[a,b]$, and differentiable on (a,b), then there is at least one point c with $a < c < b$, such that $f'(c) = \{f(b)-f(a)\}/(b-a)$.
Proof: Reduce it to Rolle`s theorem as follows: if $m = \{f(b)-f(a)\}/(b-a)$, define $g(x) = f(x) - m(x-a) - f(a)$. Then g is also continuous on $[a,b]$ and differentiable on (a,b), and $g(a) = 0 = g(b)$. Hence by Rolle's theorem, there is some c with $a < c$, b where $g'(c) = 0$. But $g'(x) = f'(x) - m$, so $f'(x) = m = \{f(b)-f(a)\}/(b-a)$. **QED.**

This allows us to prove the obvious looking but very powerful result:
Corollary 8: If f is continuous on $[a,b]$, and differentiable at least on (a,b), and if $f'(x) = 0$ for all x in (a,b), then f is constant on $[a,b]$.
proof: We will prove that f has the same value at any two points $c < d$ of $[a,b]$. If $a \leq c < d \leq b$, then apply MVT to f on $[c,d]$. Clearly f is continuous on $[c,d]$ and differentiable on (c,d), so the theorem applies. Hence there is some e with $c < e < d$ where $f'(e) = \{f(d)-f(c)\}/(d-c)$. Since $f'(x) = 0$ for all x, thus $f'(e) = 0 = \{f(d)-f(c)\}/(d-c)$, so $f(d) = f(c)$. **QED.**

Actually the version that we will use the most is this:
Corollary 9: If f and g are continuous on $[a,b]$, and differentiable at least on (a,b), and if $f'(x) = g'(x)$ for all x in (a,b), then there is a constant C such that $f(x) = g(x) + C$, for all x in $[a,b]$.
proof: This follows from corollary 8 applied to the function f-g. **QED.**

Remark: If all you want is Cor. 7, it is easier to use Rolle to prove the MVT and then the MVT to prove Cor. 7, but as remarked above, our Cors. 5 and 6 are stronger statements than Cor. 7 and are useful to know. It is also possible to prove Cor. 6 directly from the MVT (as in Courant), and then deduce our Cor.5 from Cors. 6 and 7. But I like our way of doing it, since our deduction of Cors. 5 and 6 seems easier than the proof in Courant using MVT.