

2300H Rates of change.

We know a derivative dy/dx is a limit of expressions of “difference quotients”. This means the difference quotients are approximations to the derivative dy/dx , and also that the derivative dy/dx is an approximation to the difference quotients. Thus to understand what a derivative is in some real life situations we should understand what difference quotients are. Then derivatives are approximations to those things.

Well a difference measures a change in some quantity. If a ball has height 100 feet at one moment and height 10 feet a little later, then the difference $100-10 = 90$ feet measures the change in height of the ball. If the time when it had height 100 feet was 12 noon, and the time at which it had height 10 feet was 12 noon plus 30 seconds, then the change in time was 30 seconds. If we divide those differences, we get $90/30$ “feet per second”, which measures how fast the height of the ball was changing, on average, during that 30 second time period.

If we measure the height also at 12 noon plus 15 seconds, and also at 12 noon plus 5 seconds, and also at more and more shorter time intervals past noon, and make these average calculations again and again, the limit as the time interval goes to zero, gives the derivative of the function $h(t)$ i.e. the derivative of height with respect to the time variable. This derivative might be called the “instantaneous” rate of change of the height at the instant when it was exactly noon. This is a kind of nonsense language, since precisely at that instant there is of course no change at all in the height, but what it means is that the derivative is a good approximation to the average rate of change of height in any short time interval right after noon.

If on the other hand $y(t)$ is the value in dollars of your retirement portfolio at time t , then the derivative dy/dt at any time t , is an approximation to how rapidly your stocks were changing in value over short time intervals around that time. For example if you owned a lot of technology stocks, you know the derivative of that function was very positive in February 2000 but very negative in March and April. If $V(t)$ is the volume of water in a pool at time t , then the derivative dV/dt is a measure of how fast the pool is filling up at any instant. For example if dV/dt is negative at a given time, then the pool loses water during any short interval around that time. So if $f(t)$ is any function where t represents time, then the derivative $f'(t_0)$ is an approximation to the average rate at which the values of f are changing in short intervals of time around the time t_0 .

Now we know the derivative of a function of time is called a rate. For example the derivative of position with respect to time is called “velocity”. What about the derivative of velocity with respect to time? So this derivative approximates the change in velocity over time. Thus if $v(t)$ is the velocity of an object at time t , and $v'(t)$ is positive, it means the velocity is increasing in intervals around that instant, i.e. the object is speeding up at that instant. This is caused as we learn in physics by applying a force to the object. Of course you cannot always see a force (think of gravity), so we just observe the object and measure its velocity, and if the velocity changes we conclude there is a force acting on the object. An increase in velocity is called an acceleration. More precisely acceleration is change in velocity, and if the acceleration at a given time is zero, it means the average velocity is roughly the same, in all small intervals around that time.

We can also speak of accelerating the flow of water into the pool, hence the derivative $V'(t)$ of

the volume of water in the pool gives the rate at which the pool is being filled, and the derivative $V''(t)$ of that derivative, measures the increase or decrease in that rate of flow. I.e. if someone said the pool is not being filled fast enough and we start opening the valves wider to fill it faster, that should be measured by a positive value of $V''(t)$. So $V'(t)$ measures speed of flow and $V''(t)$ measures acceleration of that flow. So we tend to call a derivative with respect to time a rate, and a second derivative with respect to time (i.e. the derivative of a derivative), an acceleration.

Some derivatives with respect to other quantities also have meaningful interpretations. When my brother was an engineering student he told me of a wonderful number they called the “fudge factor”. This number when multiplied by your answer always gave the right answer. How to find it? Divide the right answer by your answer of course. Now the derivative is a little like this only better. Since $dy/dx = y'$ is a good approximation to the difference quotient $(y-y_0)/(x-x_0)$, then the product $y'(x-x_0)$ is a good approximation to $y-y_0$. So the derivative y' is a number which when multiplied by the change in x gives a good approximation to the change in y . And, unlike the fudge factor, you don't already have to know the change in y to calculate it either. This concept lets us interpret a lot of derivatives.

For example, if you are looking at the area function for a square in terms of the base side, $A(s) = s^2$. Then the derivative $dA/ds = 2s$, is a number which when multiplied by the change in the side $s-s_0$, gives an approximation to the change in area $A-A_0$. If you draw a picture of a square with side s and another larger square with side $(s + (s-s_0))$, you will see roughly how much of the change in area you get by the approximation $(2s)(s-s_0)$. So in this case the derivative of the area with respect to the length is (twice) a length. If you draw a picture of this situation you will also see how far off this approximation is, namely it will give you only the part of the change in area which is linear in $(s-s_0)$. I.e. enlarging the side of a square by $s-s_0$, enlarges the area by two rectangles and one small square, and multiplying the derivative by $(s-s_0)$ only gives you the area of the two rectangles.

If you consider $A(r) = \pi r^2$, the area function of a circle with respect to the radius, then the derivative $dA/dr = 2\pi r$, is a number which when multiplied by $r-r_0$, approximates the change $A-A_0$ in area of the circle. Notice that this is just the circumference of the circle. If you draw a circle of radius r and another larger one of area $r + r-r_0$, you will see why the change in area is approximated by the product $(2\pi r)(r-r_0)$. Thus in this case the derivative of the area of a circle with respect to the radius (a length) is the circumference of the circle (another length).

If you consider the area function of a sphere, $A = (4/3)\pi r^3$, with respect to radius, you may know that the derivative $dA/dr = 4\pi r^2$, is the area of the sphere. Can you picture a sphere of radius r and another one of radius $r + r-r_0$ well enough to see why the increase $A-A_0$ in area is approximated by the product $(4\pi r^2)(r-r_0)$? In this case the derivative of a volume with respect to radius (a length) was area. If you let $V(s) = s^3$ be the volume of a cube, with respect to the side s of the cube then the derivative $dV/ds = 3s^2$, is (3 times) the area of a side, another area like quantity.

Consider the graph $y = f(x)$ of a positive valued function, and the area bounded between that

graph, the x axis, the y axis, and the vertical line passing through the point x on the horizontal axis. This area $A(x)$ is a function of x , i.e. as we move the vertical line further to the right we enclose more area. Draw the picture of this area for x and for $x+(x-x_0)$ and compare the two areas. Does it look as if $A-A_0 = A(x+x-x_0) - A(x)$ is roughly $f(x).(x-x_0)$? If so, that seems to indicate that the derivative dA/dx is $f(x)$. This gives a way to construct a function $A(x)$ whose derivative is your original function $f(x)$. This process of going from a given function to its area function is often called "integration", and the fact that the resulting area function has the original function as its derivative (at least under the hypothesis that the original function was continuous) is called the "fundamental theorem of calculus".