**Exponential and logarithmic functions**

We are concerned with making sense of expressions like $a^b$ where a and b are real numbers, and with deciding just what $a^b$ should mean. We start by letting a be a positive real number, say 2. Then we know of course that when n is a positive integer, $2^n$ is a "power" of 2, or 2 multiplied by itself n times. I.e. $2^1 = 2$, $2^2 = 2(2)$, $2^3 = 2(2^2)$, and more generally, for any positive integer k, $2^k = 2(2^{k-1}) = (2)(2)(.....)(2)$ = a product of k factors of 2.

If we want to define $2^x$ where x is a fraction such as $x = 1/2$, or if $x = 0$, or $x = -3$, then we have to decide what we want to be true about expressions like $a^b$ and use this to guide us in our choice of definition.

The most fundamental property turns out to be this: whenever n,m are two positive integers, we always have $a^{(n+m)} = a^n.a^m$. This is called the "homomorphism" law for exponents.

We can prove this by induction as follows: let m be any positive integer, then we want to prove that $a^{(n+m)} = a^n.a^m$, for every positive integer n. The principle of induction says that if we can prove this for n = 1, and if we can also prove that whenever it is true for one value of n, then it is also true for the next, then it must be true for all values of n.

I.e. if we prove it for 1, then it is also true for the next value of n, so it is also true for 2. Then since it is true for 2 it must also be true for the next value so it is true for 3, and so on. In this way it must be true for all values of n.

So we must prove two things:
1) that our formula is true for n = 1, and
2) whenever it is true for one value of n, it is then also true for the next value.

We start by noting that for n = 1, the rule $a^{(1+m)} = a^1(a^m)$ is true simply by definition of power notation, i.e. we defined $a^k$ to mean $a(a^{k-1})$, for every positive integer k, so taking k = m+1, we get that $a^{(1+m)} = a^1(a^m)$ is true simply by definition.

Now assume that the rule $a^{(n+m)} = a^n.a^m$ is true for some value of n, and we want to prove it must also be true for the next value, i.e. for n+1. Thus we are assuming that $a^{(n+m)} = a^n.a^m$, and we want to prove that $a^{(n+1+m)} = a^{n+1}.a^m$. But by definition, $a^{(n+1+m)} = a(a^{n+m})$, which by assumption equals $a(a^n a^m)$, which by associativity of multiplication equals $(a.a^n)a^m$, which by definition of power notation, equals $a^{n+1}.a^m$.

Thus indeed assuming that $a^{(n+m)} = a^n.a^m$ is true, leads to the conclusion that $a^{(n+1+m)} = a^{n+1}.a^m$ is true also. Hence $a^{(n+m)} = a^n.a^m$ is true for all positive integers n,m. Conversely if we want this law to hold, then just knowing that $a^1 = a$, forces $a^2 = a^1a^1 = a.a$, and $a^3 = a^1a^2 = a.a.a$, etc,..., $a^n$ must equal (a.a.....a), a product of n factors of a.

Thus any function f with the property f(1) = a and f(n+m) = f(n).f(m) must satisfy $f(n) = a^n$ for all positive integers n.

Now this property of exponentiation is so useful that we do not want to give it up. So if we want to extend the definition of exponentiation to include fractional and negative exponents, we will try to do so in a way that keeps this property true. It turns out there is one and only one way to do this, for all rational exponents.

For example if we want to define $a^0$ we would want to have $a = a^1 = a^{1+0} = a^1.a^0 = a.a^0$, so $a^0$ must be a number which multiplies a into itself. The only number that does this is 1, so we MUST take $a^0 = 1$.

If we want to define $a^{-n}$ where n is a positive integer then we want the rule $a^n a^{-n} = a^{n-n} = a^0 = 1$, so we must define $a^{-n} = 1/a^n$, which is possible only if $a \neq 0$. Then to define $a^{1/2}$ we want $a^{1/2}a^{1/2} = a^{1/2+1/2} = a^1 = a$, so we must have $a^{1/2} = \pm$ sqrt(a), and thus a must be positive, and then it is natural to take $a^{1/2} = +$ sqrt(a).

Similarly we must take $a^{1/3}$ = cuberoot(a), and $a^{1/n}$ = nth root(a). Then for n,m integers and m > 0, we must take $a^{n/m}$ = (mth root(a))$^n$. Thus the definition of $a^{n/m}$ is entirely forced upon us when we simply say that $a^1 = a$, and $a^{x+y} = a^x a^y$
.
Now when a > 1, the function we have defined on rational numbers n/m is increasing since for all m > 0 the mthroot(a) > 1, so also for all n,m, with m > 0, we have $a^{n/m} > 1$ also. Thus $a^x$ is increasing, since if n/m > s/t, then (n/m - s/t) = r > 0 where r is rational and positive so $a^{n/m} = a^{(s/t + r)} = a^{s/t} \backslash a^r > a^{s/t}$.

Now since $a^x$ is increasing for rational x, we can define $a^x$ for irrational x so as to keep it increasing. I.e. if x is irrational and r is rational and r < x, then we must have $a^r < a^x$. But if r is very close to x and r < x then to make $a^x$ a continuous function we must ahve $a^r$ very close to $a^x$, so we define $a^x = \text{glb}\ \{a^r$ for all rational $r \leq x\}$.

Similarly if a < 1 we take $a^x = \text{glb}\ \{a^r$ for all rational $r \leq x\}$. To check then that $a^x$ defined this way is continuous, we would have to check that also when a > 1, that $a^x = \text{glb}\ \{a^r$ for all rational $r \geq x\}$, i.e. that if we approximate x from above we get the same value for $a^x$ as when we approximate x from below. This is easy to check, using the fact that as k -->∞, $a^{1/k}$ -->1, since then if n/m < x < n/m + 1/k, then $a^{n/m}$ is very close to $a^{(n/m + 1/k)}$, hence there is so little room between $a^{n/m}$ and $a^{(n/m + 1/k)}$ that there is no choice for what $a^x$ must be if we are to have $a^{n/m} < a^x < a^{(n/m + 1/k)}$. This also forces continuity of the function $a^x$. Then since addition and multiplication are continuous too, and since the law $a^{x+y} = a^x.a^y$ holds for all rational x,y, it holds for all real x, y too.

Since the laws $a^1 = a$, $a^{x+y} = a^x.a^y$, plus continuity of the function $a^x$, forced the choice of all the values of the function, no other function has these properties. I.e. we have essentially proved the following thorem.

**Theorem:** If f:R-->R is any continuous function such that $f(1) = a > 0$, and $f(x+y) = f(x)f(y)$ for all real x,y, then $f(x) = a^x$ for all real x.

It can also be shown that then $(a^x)^y = a^{xy}$ for all real x,y.
Next we want to show $a^x$ is differentiable and the following partial result is easy to prove.
**Theorem:** If $a > 0$, then $a^x$ is differentiable at one point if and only if it is differentiable at all points, if and only if it is differentiable at $x = 0$, if and only if the limit $\lim_{h\to 0} \frac{a^h - 1}{h} = k$, exists and is finite. if this limit exists then $d(a^x)/dx = k\, a^x$. I.e. if $a^x$ is differentiable, then the derivative is a constant times $a^x$.
**proof:** $d(a^x)/dx = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \lim_{h\to 0} a^x (\frac{a^h - 1}{h}) = a^x \lim_{h\to 0} (\frac{a^h - 1}{h})$. Thus the limit $\lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$ exists if and only if the limit $\lim_{h\to 0} (\frac{a^h - 1}{h}) = k$ exists, and then the derivative equals $k\, a^x$. **QED.**

It is still not clear that a^x is differentiable, since we need to prove it is differentiable at x=0.
It is not hard to prove on the other hand, that if f is some differentiable function such that f(x) is never zero, and $f'(x) = kf(x)$, for some $k \neq - 0$, then f(x) satisfies the property $f(x+y) = f(x)f(y)$ for all real x,y, and hence if $a = f(1)$, then $f(x) = a^x$ for all real x. I.e. if we could find a diufferentiable function whose derivative behaves the way the derivative if a^x should behave, then that function must equal a^x. I.e. instead of starting with a^x, and proving that it is differentiable, it may be easier to work backwards, by finding a function f(x) whose derivative f'(x) equals k.f(x), and then conclude that our differentiable fucntion equals a^x.

Still we do not know how to produce such a function, so we look instead at the *inverse* function, the log function with base a. I.e. we have seen that for $a > 1$ the function $a^x$ is strictly increasing and positive valued. Similarly, if $0 < a < 1$, then $a^x$ is strictly decreasing but still positive valued. In both cases then there is an inverse function, called the logarithm with base a.

We claim that for any given exponential function, the corresponding logarithm must be defined for all positive reals. Equivalently it suffices to show that the exponential function assumes all positive values. I.e. since for $a > 1$, we have $a = 1 + h$ for some $h > 0$, it follows that for $n > 0$, we have $a^n = (1+h)^n = 1 + nh + .... > 1 + nh$. Thus as n -->∞, also $a^n$ -->∞. Thus $a^x$ assumes arbitrarily large positive values. Since $a^{-n} = 1/a^n$, the function $a^x$ also assumes arbitrarily small positive values, as x -->∞. Thus $a^x$ is defined at all real numbers and assumes all positive values.

Hence its inverse function, $\log_a(x)$, is defined on all positive numbers and assumes all real values.

Then we get the following corollary from the inverse function theorem.
**Theorem:** The function $a^x$ is differentiable, with derivative $(a^x)' = ka^x$, if and only if the function $\log_a(x)$ is differentiable, with derivative $(\log_a(x))' = 1/(kx)$.

I.e. if we can prove the logarithm is differentiable, it will follow that the corresonding exponential function is also differentiable.

Now the theorem characterizing exponential functions yields the following equivalent theorem characterizing logarithm functions.
**Theorem:** If g:R+-->R is a continuous function such that $g(xy) = g(x) + g(y)$, for all x,y, > 0, then g takes on all real values, in particular it takes the value 1, and if $g(a) = 1$, then $g(x) = \log_a(x)$, for all positive x.

Thus if we can find a differentiable function g:R+-->R such that $g(xy) = g(x) + g(y)$ for all $x,y > 0$, then g(x) must be $\log_a(x)$, where $g(a) = 1$, hence $g'(x) = 1/kx$, and the inverse function f(x) must be differentiable and equal to the exponential function $a^x$. In particular we would have proved that the exponential function is differentiable, since it would be the inverse of a differentiable function.

So we are looking for a differentiable function g:R+-->R such that $g(xy) = g(x) + g(y)$. Actually it is sufficient just to find a function with right derivative, for the following reason.

**Theorem:** If g:R+-->R is any differentiable function with $g(1) = 0$, and $g'(x) = 1/kx$, then we also have $g(xy) = g(x) + g(y)$ for all x,y, > 0. Thus, by the previous theorem, $g(x) = \log_a(x)$ where $g(a) = 1$.
**Proof:** Let b be any positive number and consider the function $h(x) = g(bx)$. The derivative $h'(x) = \{1/(kbx)\}.b = 1/(kx) = g'(x)$. Thus h and g differ by a constant, i.e. $h(x) = g(bx) = g(x) + C$. To see what C is, just put $x = 1$. Then $g(b) = g(1) + C, = 0 + C$, so $C = g(b)$. Thus we have proved that $h(x) = g(bx) = g(x) + g(b)$, as we wished to show. **QED.**

Thus to produce a differentiable log function we just need to find a differentiable function g with $g'(x) = 1/kx$. It turns out we can actually construct a differentiable function having ANY continuous function as its derivative. This is done means of the "area function" construction. I.e. consider the graph of the function 1/x for example, i.e. take $k = 1$. Now define for $x > 1$, the area function A(x) = the area above the x axis, between 1 and x on the x axis,and below the graph of $y = 1/x$. Then we claim that A(x) is diffrentiable and $A'(x) = 1/x$. Moreover we can define for $0 < x < 1$, A(x) = the negative of the area between x and 1, and above the x axis, and below the graph of $y = 1/x$, and then the same is true, and we have A:R+-->R. We have shown this in class.

It helps to understand what derivatives mean. We know the derivative of a height function is a slope function and the derivative of a position function is a velocity function, and the derivative of a velocity function is an acceleration function, but what is the derivative of an area function, or a volume function?

It turns out the derivative of an area function is a height function, and we can use this to construct functions with given continuous derivatives. In particular since $1/cx$ is continuous, we can construct a function with derivative equal to $1/cx$, for any c. When $c = 1$, the derivative is $1/x$ and the function is the logarithm of an exponential function with base equal to the value $e > 1$, such that the area under $1/x$ between 1 and e, equals 1. This point is at about $e \approx 2.71828$. The corresponding exponential function is $e^x$, and the corresponding logarithmic function is $\log_e(x)$, but because we want a shorter name for this log function we call it $\ln(x)$ and call it the "natural" log function. Then $e^x$ is differentiable with derivative $e^x$.

Finally we can prove $2^x$ is also differentiable as follows. Since $\ln(x) = \log_e(x)$ takes on all real values, it takes the value 2 somewhere, i.e. at the point $\log_e(2) = \ln(2)$. Hence $e^{\ln(2)} = 2$, so $2^x = (e^{\ln(2)})^x = e^{\ln(2)x}$. Thus we can differentiate $2^x$ using the chain rule. I.e. $(2^x)' = (e^{\ln(2)x})' = (e^{\ln(2)x}).\ln(2) = 2^x.\ln(2)$. So the constant $k = \ln(2)$.

Thus for any $a > 0$, we have $(a^x)' = a^x.\ln(a)$, and $(\log_a(x))' = 1/(\ln(a).x)$. Further, $a^x = e^{\ln(a)x}$, and thus $\log_a(x) = \ln(x)/\ln(a)$.