2310H Fundamental theorem of calculus

We have seen that the idea and technique of "Riemann integration" was essentially understood by Archimedes. He could even calculate the area under a parabola and the volume of a sphere as limits of Riemann sums. What he did not realize, and no one else did until over 1,000 years later, was the connection between integration and differentiation. I.e. when we allow the upper endpoint of the integral to vary, and obtain a function, the "indefinite integral", it turns out that this function is differentiable, at least as long as the integrand is continuous, and the derivative of the indefinite integral is the original integrand. This statement is called the fundamental theorem of calculus, as its links the two branches of calculus together, and unleashes their full power.

We will give a slightly more precise statement that applies also when the integrand is not everywhere continuous, but merely integrable, with a view to preparing you for more advanced courses where these ideas will be more fully explained and used. In order not to raise the difficulty level too much however we will omit proofs of more general statements when those proofs are harder than the usual ones. Proofs will be gladly supplied on request.

First we state and prove some useful properties of Riemann integrals. We remark that proofs of theorems about integration can be notoriously detailed and picky. So do not worry if you do not understand every little argument. Feel free to ask about them however. For example, to me the proof of part (i) below is very techniucal, and not hugely interesting. The statement of (i) however, is absolutely fundamental for the proof of the fundamental theorem of calculus, so we must at least know that. At the moment I cannot think how to prove the first part of (iv) myself, at least not without using more advanced facts that we have not proved. In class yesterday I only proved parts (ii) and (iii), which seemed relatively easy to explain. I am not trying to prove everything in this course, only enough to give the flavor of what the proofs are like. Too many proofs makes the course too technical. On the other hand no proofs at all would make it superficial. However in these notes I will give more proofs, at least if they seem to use only ideas that we know. If I were you I would skip the proofs of this next "technical theorem" on first reading anyway. Get on to the good stuff after it, the FTC.

Technical Theorem (elementary properties of integrals):
(i)(interval subdivision)
Let f be a function defined on the closed bounded interval [a,b] and let c be a point such that a< c < b. Then f is integrable on the interval [a,b] if and only if f is integrable on both intervals [a,c] and [c,b]. Moreover in this case, $\int_a^b f = \int_a^c f + \int_c^b f$.

(ii)(linear combinations)
If f and g are both integrable on the interval [a,b], and c is a constant, then cf and f+g are also integrable on the interval [a,b]. Moreover in this case, $\int_a^b cf = c\int_a^b f$, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(iii)(monotonicity)

If f and g are both integrable on the interval [a,b] and if $f(x) \leq g(x)$ for every x in [a,b], then $\int_a^b f \leq \int_a^b g$.

(iv) If f is integrable on the interval [a,b], then so is |f|, and $|\int_a^b f \;| \leq \int_c^b |f|$.

proof(i): Assume f is integrable on both [a,c] and [c,b]. By one of our criteria for integrability, to prove f integrable on [a,b] it suffices to show, for any given e>0, that we can find upper and lower sums U,L, for f on [a,b] such that U-L < e. If you think about the meaning of such upper and lower sums you will see that putting two upper sums together, one for f on [a,c], and one for f on [c,b], yields an upper sum for f on [a,b]. The same holds for lower sums. Now since f is integrable on both [a,c] and [c,b], we can find upper and lower sums there as close together as we want, so we choose upper and lower sums U_1,L_1 on [a,c] with $U_1-L_1 < e/2$. We do the same for f on [c,b] choosing such sums U_2, L_2 with
$U_2-L_2 < e/2$. Then by combining these, we get an upper sum U and lower sum L for f on [a,b] with $U-L = (U_1-L_1) + (U_2-L_2) < e/2 + e/2 = e$.

This same argument shows that for every pair of lower sums L_1,L_2, for f on [a,c] and [c,b], we get a lower sum L for f on [a,b] with $L = L_1+L_2$. Thus sany upper bound for the lower sums of f on [a,b] must be larger than the sum of any two lower sums L_1 and L_2 for f on [a,c] and [c,b]. Thus the integral for f on [a,b] is at least as large as $\int_a^c f \;+ \int_c^b f$.

Now assume f integrable on [a,b], and prove it is also integrable say on [a,c]. Given e > 0, just choose upper and lower sums U,L, for f on [a,b] with U-L < e. Then include c as a subdivision point, if it was not already. This makes a slightly more refined subdivision with the same upper and lower sums. Then choose only that part of the subdivision from a to c, as a subdivision for f on [a,c], and use the same upper and lower bounds for f. This gives upper and lower sums U_1, L_1 for f on [a,c] with difference even smaller than it was before, in particular less than e. A similar argument works for [c,b]. Indeed one sees as well that the two lower sums L_1, L_2 constructed in this way satisfy $L_1 + L_2 = L$. Hence the set of lower sums for f on [a,b] is contained in the set of sums of lowers sums L_1+L_2 for f on [a,c] and [c,b]. Thus the number $\int_a^c f \;+ \int_c^b f$ is at least as large as $\int_a^b f$. This proves all of (i).

proof of (ii): We use the other version of integrability this time, limits of Riemann sums. As in class, we want to see if the limit of the sums
$\sum_{i=1}^{n}(f+g)(\xi_i)\Delta x_i$ exists as all Δx_i—>0. But this can easily be broken up into two sums, both of which do have limits, and so this limit exists and is the sum of those limits, i.e. $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. The other one is an exercise. this proves (ii).

proof of (iii). Every lower sum for f on [a,b] is also a lower sum for g on [a,b]. Hence the integral of g, which is $\geq$ all lower sums for g , is also $\geq$ all lower sums for f. But by definition of the integral of f as the smallest number with that proerty, we get $\int_a^b f \leq \int_a^b g$.

[Notice, yesterday I chose to prove the ones with the shortest proofs. Or maybe these proofs are just shorter now because I practiced writing them in class, and now they have become shorter the second time.]

proof of (iv) Well, the easiest proof (for me) that f integrable implies that |f| is also integrable, is to use the fact we stated but did not prove, that a function is Riemann integrable if and only if it is bounded, and discontinuous at most on a set of "measure zero". Then since taking absolute value makes f more continuous, the set of discontinuities of |f| is smaller than that of f, hence still of measure zero. So |f| is also bounded and more continuous than f was, hence |f| is also integrable. But there must be some elementary argument. Does anyone see one?

Assuming that, the inequality in (iv) follows from the monotonicity property (iii). I.e. we have always $-|f| \leq f \leq |f|$, so the integrals satisfy the analogous inequalities: $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$.
But $-A \leq B \leq A$, just means that $|B| \leq A$. So we get $|\int_a^b f| \leq \int_a^b |f|$. QED.

[It was a good exercise for me to write these notes however because I had not even noticed that the first part of (iv) needed proof.]

Probably some nonsense with upper and lower sums will work. Lets try that. Excuse for a minute or two of thought with paper and pen..... Ok, a few pictures sufficed to persuade me it is indeed true and elementary, but I will not write all out in detail. The point is that if we have an upper and lower sum U,L, for f on [a,b] with U-L < e, then we can immediately produce an upper and lower sum U_1,L_1 for |f| on [a,b] with $U_1-L_1 < e$ also. I.e. notice that U-L is just the sum of the areas of some rectangles that entirely contain the grahh of f, one rectangle over each subinterval. I.e. if $m_i \leq f(x) \leq M_i$ for all x in $[x_{i-1},x_i]$, let R_i be the rectangle with base parallel to and directly over (or under) the subinterval $[x_{i-1},x_i]$, and with lower edge at height m_i and upper edge at height M_i. Then R_i completely contains the graph of f over $[x_{i-1},x_i]$, and R_i has area $(x_i - x_{i-1})(M_i-m_i)$. It is the sum of the areas of these R_i that is by assumption less than e.

Ok, now consider three cases for the numbers m_i,M_i. If both are non negative, then the function f has only non negative values on the interval $[x_{i-1},x_i]$, so also does |f|, and the same m_i and M_i work also for |f|, and we get the same rectangle. I.e. then also $m_i \leq |f| \leq M_i$ on $[x_{i-1},x_i]$, and we have part of an upper and lower sum for |f|. On the other hand if both m_i and M_i are non positive, then since $m_i \leq f(x) \leq M_i$ on $[x_{i-1},x_i]$, and all these numbers are negative, when we take absolute values it reverses everything, and we get $-m_i \geq |f(x)| \geq -M_i$ on $[x_{i-1},x_i]$, so $-m_i$ is an upper bound for |f| on $[x_{i-1},x_i]$, and $-M_i$ is a lower bound. But the rectangle they form has the same area as before. Finally if $M_i > 0$ and $m_i < 0$, then the values of |f| are caught between 0 and the larger of M_i or $-m_i$. Either way the rectangle gets smalller. Thus the resulting upper and lower sums for |f| are closer together than were the ones for f, so the difference is still less than e. Hence f integrable does imply |f| also integrable. **QED.**

Now we are ready for the fundamental theorem describing the deeper, less expected properties of the integral, i.e. properties of the indefinite integral function.

Theorem: Let f be integrable on [a,b] and define the function G(x) = $\int_a^x f$, called the "indefinite integral" of f.
(i) The function G(x) is Lipschitz continuous on [a,b]; in particular if K is a number such that $|f(x)| \le K$ for all x in [a,b], then K is a Lipschitz constant for G on [a,b].
(ii) If f is continulous at a point c in [a,b], then G is differentiable at c, and G'(c) = f(c).
Proof:(i) Since f is integrable on [a,b] f is bounded so there is some number K with $|f(x)| \le K$ for all x in [a,b]. We will show that K is a Lipschitz constant for G on [a,b]. I.e. we will show that for all pairs of points c,d with $a \le c \le d \le b$, that $|G(d)-G(c)| \le K|d-c|$. For this notice that using the previous technical theorem we have G(d)-G(c) = $\int_c^d f$. Hence, using other technical properties, $|G(d)-G(c)| = |\int_c^d f \,| \le \int_c^d |f| \le \int_c^d K$ = K(d-c) = K|d-c|. That proves (i).

proof of (ii) We want to show the limit of $\frac{G(x)-G(c)}{x-c}$ exists and equals f(c), if f is continuous at c. Assuming that, and given e>0, then we can find d >0 such that whenever |x-c| < d, then we have f(c)-e < f(x) < f(c)+e, i.e. |f(x)-f(c)| < e.

Hence |x-c| < d also implies (where we assume for simplicity that $c \le x$):

$$(f(c)-e)(x-c) = \int_c^x (f(c)-e) \le \int_c^x f \le \int_c^x (f(c)+e) = (f(c)+e)(x-c).$$ That is,

$$f(c)(x-c) - e(x-c) \le \int_c^x f \le f(c)(x-c) + e(x-c),$$ i.e.

$$\left|\int_c^x f - f(c)(x-c)\right| \le e(x-c),$$ when |x-c| < d.

We want to show that $\left|\frac{G(x)-G(c)}{x-c} - f(c)\right|$ is also less than e. Since G(x)-G(c) = $\int_c^x f$, we get

$$\left|\frac{G(x)-G(c)}{x-c} - f(c)\right| = \left|\frac{\int_c^x f}{x-c} - f(c)\right|$$

$$= \left|\frac{\int_c^x f - f(c)(x-c)}{x-c}\right| \le e|x-c|/|x-c| = e.$$ Hence, (after one does the argument also for the case x < c), we have that G'(c) = f(c). **QED.**

It is late and I don't have time to proofread this so let me know if it seems wrong.