Hamiltonian, and operator-definition of angular momentum for central potential,

$$H = \frac{p^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \frac{\hbar}{i}\nabla \qquad \text{[I.1]}$$

Some commutators that are easy to see given the canonical $[r_i, p_j] \equiv \mathbf{i}\hbar \cdot \delta_{ij}$,

$$[L_x, y] = [yp_z - zp_y, y] = -z[p_y, y] = i\hbar z \qquad [L_x, p_y] = [yp_z - zp_y, p_y] = [y, p_y]p_z = i\hbar p_z \qquad [L_z, x] = 0, \qquad [L_x, p_x] = 0 \qquad \text{[I.2]}$$

$$[L_x, L_y] = [L_x, zp_x - xp_z] = [L_x, z]p_x - x[L_x, p_z] = -i\hbar y p_x + i\hbar x p_y = i\hbar L_z \qquad \text{[I.3]}$$

By cyclic permutation of [I.3], we can deduce,

$$[L_i, L_j] \equiv \mathbf{i}\hbar L_k \quad \leftrightarrow \quad \mathbf{L} \times \mathbf{L} = \mathbf{i}\hbar \cdot \mathbf{L} \qquad \text{[I.4]}$$

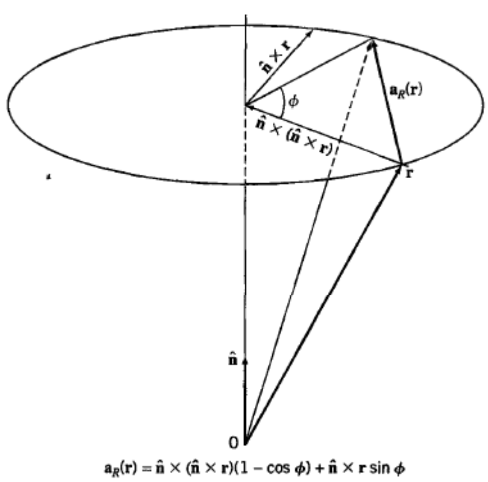
Analogy with rigid rotations

Consider f to be a function of a vector. It's mathematics, not quantum mechanics. Then, finite and infinitesimal translations are effected by linearity (Taylor series),

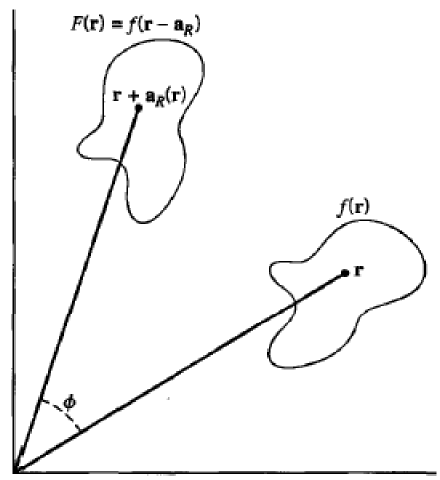
$$f(\mathbf{r}) \mapsto F(\mathbf{r} + \mathbf{a}) = f(\mathbf{r}) \qquad F(\mathbf{r} + \boldsymbol{\varepsilon}) = F(\mathbf{r}) + \boldsymbol{\varepsilon} \cdot \nabla F(\mathbf{r}) = f(\mathbf{r}) \qquad \text{[I.5]}$$

A finite *rotation*, on the other hand, is a bit more complicated. Recall QM 08, 320 - pr 4-3 - generator of arbitrary rotations in R3 and/or CM 04, 181 - de 12 - the rotation formula, where we derived,

$$\mathbf{r} \to \mathbf{r}' = \mathbf{n} \times (\mathbf{n} \times \mathbf{r})(1 - \cos\phi) - \mathbf{n} \times \mathbf{r} \cdot \sin\phi = \mathbf{a}_R \qquad \text{[I.6]}$$



(a)



(b)

[I.7]

Rotation about an axis defined by the unit vector $\hat{\mathbf{n}}$ and the rotation angle ϕ (a) shows the displacement $\mathbf{a}_R$ (see [I.6]) of the point whose position vector is r. (b) illustrates the active rotation of a function or state f(r) about an axis ($\hat{\mathbf{n}}$) perpendicular to the plane of the figure: $f(\mathbf{r}) \to F(\mathbf{r}) = f(\mathbf{r} - \mathbf{a}_R)$.

Verify $\mathbf{r} \to \mathbf{r}' = \mathbf{n} \times (\mathbf{n} \times \mathbf{r})(1 - \cos\phi) - \mathbf{n} \times \mathbf{r} \cdot \sin\phi = \mathbf{a}_R$.

See QM 08, 320 - pr 4-3 - generator of arbitrary rotations in R3 and/or CM 04, 181 - de 12 - the rotation formula.

Show that it gives the expected answer for a rotation about the z-axis. Using $\hat{z} \times (\hat{z} \times (x, y)) = \hat{z} \times (+y, -x) = (+y, -x)$, an arbitrary vector (x,y) in the xy-plane is rotated as,

$$(x,y)\to(x',y')=\hat{z}\times(\hat{z}\times(x,y))(1-\cos\phi)-\hat{z}\times(x,y)\cdot\sin\phi=\hat{z}\times(+y,-x)(1-\cos\phi)-(+y,-x)\cdot\sin\phi$$
$$=\begin{bmatrix}-x\\-y\end{bmatrix}(1-\cos\phi)-\begin{bmatrix}y\\-x\end{bmatrix}\sin\phi=\begin{bmatrix}x\\y\end{bmatrix}\cos\phi+\begin{bmatrix}-y\\x\end{bmatrix}\sin\phi-\begin{bmatrix}x\\y\end{bmatrix}=\boxed{\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}x\\y\end{bmatrix}} \qquad [I.1]$$

Mistake? I think I'm off by a minus sign in [I.1]. I think the – [x,y] should be a +[x,y] (the very last column-vector that looks like a translation added onto the rotation we derived, in [I.1]. If this was a +, it would look a *bit* like the end I'm working towards in Shankar 12.4.4, with the cosines and sines constituting broken-up components of $\vec{\theta}$ that is supposed to appear in a cross product).