*Recall that* **V** *is a vector operator if its operator components V[i] transform as,*

$$U^{\dagger}[R]V_i U[R] = R_{ij}V_j \qquad \text{[I.1]}$$

*1) for an infinitesimal rotation* $\delta\vec{\theta}$ *show, on the basis of the previous exercise, that,*

$$R_{ij}V_j = V_i + (\delta\vec{\theta}\times\mathbf{V})_i = V_i + \varepsilon_{ijk}(\delta\theta)_j V_k \qquad \text{[I.2]}$$

See what the RHS of [I.1] implies.

$$R_{ij}V_j = U^{\dagger}(\delta\vec{\theta})V_i U(\delta\vec{\theta}) = (\mathbf{1} - \tfrac{1}{i\hbar}\delta\vec{\theta}\bullet L_{\hat{\theta}})V_i(\mathbf{1} + \tfrac{1}{i\hbar}\delta\vec{\theta}\bullet L_{\hat{\theta}}) = V_i + \tfrac{1}{i\hbar}[V_i, \delta\theta_j L_j] + \mathcal{O}(\delta\theta^2) = V_i + \tfrac{1}{i\hbar}[V_i, \delta\theta_j(R_k P_i - R_i P_k)]$$
$$= V_i + \tfrac{1}{i\hbar}[V_i, \delta\theta_j(R_k P_i - R_i P_k)] = \ldots = V_i + \varepsilon_{ijk}(\delta\theta)_j V_k \qquad \text{[I.3]}$$

Another try

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Oops…I think this carries implications for part (2) of this problem (below). We should work with the LHS of [I.1]. The R[ij] looks like it introduces the notion of a "cross product" by looking at [I.2]. Expand it out,

$$R_{ij}V_j = V_i + \varepsilon_{ijk}(\delta\theta)_j V_k = V_i + (\delta\theta_j V_k - \delta\theta_k V_j) \rightarrow \mathbf{V}' = \mathbf{V} + \text{oops....same cross product: tautology} \qquad \text{[I.4]}$$

Try using the general rotation matrix by the Euler angles for the angles → d(theta),

$$RZ1 := \begin{bmatrix} 1 & x1 & 0 \\ -x1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad RY2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x2 \\ 0 & -x2 & 1 \end{bmatrix} \quad RZ3 := \begin{bmatrix} 1 & x3 & 0 \\ -x3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have a X-rotate, Z-rotate, X-rotate. Composing them,

RZ3.RY2.RZ1= matrix multiplication =

$$\begin{bmatrix} 1 - x3\,x1 & x3 + x1 & x3\,x2 \\ -x1 - x3 & 1 - x3\,x1 & x2 \\ x2\,x1 & -x2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \theta_3 + \theta_1 & 0 \\ -\theta_3 - \theta_1 & 1 & \theta_2 \\ 0 & -\theta_2 & 1 \end{bmatrix} + \mathcal{O}(\theta_i^{\,2})$$

But now, if I multiply this rotation matrix by a vector **V**, I don't see Levi-Civita-like terms appearing [(ij – ji), e.g.], which I could collect into a cross product which appears on the RHS of [I.2]…

Another another try

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Build up the matrix elements I "want" to see in order to get a cross-product,

$$R\mathbf{V} = \begin{bmatrix} 1 & +\theta_3 & -\theta_2 \\ -\theta_3 & 1 & +\theta_1 \\ +\theta_2 & -\theta_1 & 1 \end{bmatrix}\mathbf{V} = \mathbf{V} + (V_2\theta_3 - V_3\theta_2)\hat{x}_1 + (V_3\theta_1 - V_1\theta_3)\hat{x}_2 + (V_1\theta_2 - V_2\theta_1)\hat{x}_3 = \mathbf{V} + (\vec{\theta}\times\mathbf{V}) \qquad \text{[I.5]}$$

….Should this matrix in [I.5] look familiar?

Looking at the solution of a similar problem

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*Show that under an infinitesimal rotation by an angle* $\delta\vec{\theta}$ *, the vector* $\mathbf{V}$ *transforms as,*

$$\mathbf{V} \to \mathbf{V}' = R\mathbf{V} = \mathbf{V} + \delta\vec{\theta} \times \mathbf{V} \quad \leftrightarrow \quad V_j \to V_j' = R_{ij}V_j = V_i + \delta\theta_j V_k \varepsilon_{ijk} \tag{I.6}$$

**Special case:** let the z axis be along the direction of $\delta\vec{\theta}$ . In this case, $\delta\vec{\theta} = |\delta\theta|\hat{k} = \delta\theta_z\hat{k}$ . Then,

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \to \begin{bmatrix} V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} 1 & -\delta\theta_z & 0 \\ +\delta\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \mathbf{V} + \delta\theta_z(V_x - V_y) \to \boxed{\mathbf{V} = \mathbf{V}' = \mathbf{V} + \delta\vec{\theta} \times \mathbf{V}} \tag{I.7}$$

The last step was made because this is a vector equation it applies to all coordinate systems. **But I say:** You only have theta_z running around after you do the z-special case. Shouldn't you have theta_x and –theta_y running around to constitute the requested cross product?

*2) feed in* $U[R] = \mathbf{1} + \frac{1}{\mathbf{i}\hbar}\delta\vec{\theta} \bullet \mathbf{L}$ *into the LHS of* [I.1], *and thereby deduce that,*

$$\left[V_i, L_j\right] = \mathbf{i}\hbar \cdot \varepsilon_{ijk} V_k \tag{I.8}$$

$$U^\dagger[R]V_iU[R] = \left(\mathbf{1} + \mathbf{i}\tfrac{1}{\hbar}\delta\vec{\theta} \bullet \mathbf{L}\right)V_i\left(\mathbf{1} - \mathbf{i}\tfrac{1}{\hbar}\delta\vec{\theta} \bullet \mathbf{L}\right) = V_i + \tfrac{\mathbf{i}}{\hbar}\left[\delta\vec{\theta} \bullet \mathbf{L}, V_i\right] + \mathcal{O}((\delta\theta \bullet \mathbf{L})^2) = V_i + \varepsilon_{ijk}(\delta\theta)_j V_k \tag{I.9}$$

Discarding the $\mathcal{O}((\delta\theta \bullet \mathbf{L})^2)$ and absorbing a negative sign into $[A, B] \equiv -[B, A]$, we immediately conclude that,

$$\left[V_i, \delta\vec{\theta} \bullet \mathbf{L}\right] = \left[V_i, L_j\right] = \mathbf{i}\hbar \cdot \varepsilon_{ijk}(\delta\theta)_j V_k \tag{I.10}$$

Oops. We need $\delta\vec{\theta} \bullet \mathbf{L} \overset{?????}{=} L_j$, but we're pretty close…

*This is as good a definition of a vector-operator as* [I.1]. *by setting* **V = L**, *we can obtain the commutation-rules among the L's.*

For help, do sakurai problem 3.7…

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$$U^\dagger[R]V_iU[R] = R_{ij}V_j$$

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*Show that under an infinitesimal rotation by an angle* $\delta\vec{\theta}$ *, the vector* $\mathbf{V}$ *transforms as,*

$$\mathbf{V} \to \mathbf{V}' = R\mathbf{V} = \mathbf{V} + \delta\vec{\theta} \times \mathbf{V} \quad \leftrightarrow \quad V_j \to V_j' = R_{ij}V_j = V_i + \delta\theta_j V_k \varepsilon_{ijk} \tag{I.11}$$

**Special case:** let the z axis be along the direction of $\delta\vec{\theta}$ . In this case, $\delta\vec{\theta} = |\delta\theta|\hat{k} = \delta\theta_z\hat{k}$ . Then,

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \to \begin{bmatrix} V'_x \\ V'_y \\ V'_z \end{bmatrix} = \begin{bmatrix} 1 & -\delta\theta_z & 0 \\ +\delta\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \mathbf{V} + \delta\theta_z (V_x - V_y) \to \boxed{\mathbf{V} = \mathbf{V}' = \mathbf{V} + \delta\vec{\theta} \times \mathbf{V}} \qquad \text{[I.12]}$$

The last step was made because this is a vector equation it applies to all coordinate systems. **But I say:** You only have theta_z running around after you do the z-special case. Shouldn't you have theta_x and –theta_y running around to constitute the requested cross product?