

Consider the following as given,

$$\int_{-1}^{+1} P_{\ell}(x)P_{\ell'}(x) \cdot dx \equiv \frac{2}{2\ell+1} \delta_{\ell\ell'} \quad [\text{I.1}]$$

$$P_{\ell}(x) \equiv \frac{1}{2^{\ell} \ell!} \frac{d^{\ell}((x^2-1)^{\ell})}{dx^{\ell}} \quad [\text{I.2}]$$

$$\int_0^1 (1-x^2)^m dx \equiv \frac{(2m)!!}{(2m+1)!!} \quad [\text{I.3}]$$

Prove the relation,

$$e^{ikr \cos \theta} \equiv \sum_{\ell=0}^{\ell \rightarrow \infty} i^{\ell} (2\ell+1) \cdot j_{\ell}(kr) P_{\ell}(\cos \theta) \quad [\text{I.4}]$$

The Fourier expansion is,

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} C_l j_l(kr) P_l(\cos \theta) \quad [\text{I.5}]$$

So, you just need to find the coefficients. Front-multiply [I.5] by $P_{\ell'}(\cos \theta)$ and integrate,

$$\int_{-1}^{+1} P_{\ell'}(\cos \theta) e^{ikr \cos \theta} d(\cos \theta) = \sum_{\ell=0}^{\ell \rightarrow \infty} C_{\ell} j_{\ell}(kr) \int_{-1}^{+1} P_{\ell'}(\cos \theta) P_{\ell}(\cos \theta) \cdot d(\cos \theta) \quad [\text{I.6}]$$

Switching variables $x = \cos \theta$ and using [I.1], we see [I.6] becomes,

$$\int_{-1}^{+1} P_{\ell'}(x) e^{ikrx} dx = \sum_{\ell=0}^{\ell \rightarrow \infty} C_{\ell} j_{\ell}(kr) \frac{2}{2\ell+1} \delta_{\ell\ell'} = C_{\ell'} j_{\ell'}(kr) \frac{2}{2\ell'+1} \quad \leftrightarrow \quad C_{\ell'} = \frac{2\ell'+1}{2j_{\ell'}(kr)} \int_{-1}^{+1} P_{\ell'}(x) e^{ikrx} dx \quad [\text{I.7}]$$

Dropping the primes and representing $P_{\ell'}(x)$ using [I.2],

$$C_{\ell} = \frac{2\ell+1}{2j_{\ell}(kr)} \frac{1}{2^{\ell} \ell!} \int_{-1}^{+1} \frac{d^{\ell}((x^2-1)^{\ell})}{dx^{\ell}} e^{ikrx} dx \quad [\text{I.8}]$$

Um, perhaps we could the addition theorem for spherical harmonics, as physicsforums suggests?

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad [\text{I.9}]$$

Gamma is the angle between vector \mathbf{x}' and \mathbf{x} . In our case, $\mathbf{x}' = \hat{\mathbf{z}}$ and \mathbf{x} is angled at θ ; in both cases, $\phi = \phi' = 0$, so

[I.9], using also the formula $Y_{\ell}^m(\theta, \phi)^* = (-1)^m \cdot Y_{\ell}^{-m}(\theta, \phi)$, becomes,

$$P_{\ell}(\cos \theta) = P_{\ell}(x) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{m=+\ell} Y_{\ell m}^*(0,0) Y_{\ell m}(\theta, \phi) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{m=+\ell} (-1)^m \cdot Y_{\ell}^{-m}(0,0) Y_{\ell m}(\theta, 0) \quad [\text{I.10}]$$

Only the $m = 0$ spherical harmonics involve cosines, so the $\theta = 0$ terms for $m \neq 0$, which involve $\sin \theta$, disappear. We should bear in mind the formula $Y_{\ell 0}(\theta, \phi) \equiv \sqrt{\frac{2\ell+1}{4\pi}} \cdot P_{\ell}(\cos \theta)$, and [I.10] becomes,

$$P_{\ell}(\cos \theta) = \frac{4\pi}{2\ell+1} Y_{\ell}^0(0, 0) Y_{\ell 0}(\theta, 0) = \frac{4\pi}{2\ell+1} \sqrt{\frac{2\ell+1}{4\pi}} \cdot P_{\ell}(\cos 0) \sqrt{\frac{2\ell+1}{4\pi}} \cdot P_{\ell}(\cos \theta) = P_{\ell}(\cos \theta) \quad [\text{I.11}]$$

A tautology... There must be some way to use the addition theorem [I.9] that I'm not aware of.