Math 4150: Limit properties of real and complex numbers.
What are real numbers?
A positive real number is represented by an infinite decimal, i.e. a positive integer followed by a decimal point and an infinite sequence of digits between 0 and 9. We sometimes call such a decimal "finite" if eventually all digits are zero. A positive real number can have such a finite representation if and only if it is a rational number of form A/10^n, for some positive integer A and some non negative integer $n \geq 0$. Decimals with a finite representation also have another different infinite representation which ends in all 9's. E.g. 1.0 = .999999....... All other positive real numbers have only one decimal representation. We represent negative real numbers by putting a minus sign in front of a positive number. Zero is represented as 0 = 0.00000......

What arithmetic operations do they admit?
Since real numbers are infinite decimals, adding and multiplying them already presents a problem, where do you begin? Still it can be done by a limiting process. Real numbers can always be added and multiplied and subtracted, and we can divide by non zero real numbers. I.e. they form a "field".

Ordering properties
Real numbers are ordered as are the points of a line in Euclidean geometry. Given any three real numbers, exactly one of them is always between the other two, where b is between a and c if and only if $a < b < c$. Numbers greater than zero are called "positive". Ordering is preserved by adding the same thing to both sides and by multiplying by a positive number. Equivalently, sums and products of positive numbers are again positive. Thus we have $a<b$ if and only if $b-a$ is positive. Given two positive decimals in their unique infinite representation, they are ordered by lexicographic order, i.e. the larger one is the first one to have a larger entry starting from the left. E.g. 123.454567678..... is larger than 123.454567677........ It is clear from the decimal representation that the positive integers are unbounded above, i.e. given any real number A, there is a positive integer N such that $A<N$. It follows that reciprocals of positive integers are not bounded below by any positive number. I.e. given any positive number $e>0$, there is a positive integer N such that $1/N < e$. The next property of order is so important it deserves its own paragraph.

Completeness property of real numbers
An "upper bound" for a set S of real numbers is a real number B such that $x \leq B$ for all x in S, i.e. it may not be larger than everything in the set, but nothing in the set is larger than it is. The crucial property of real numbers, not possessed by rationals is this:
Every non empty set of real numbers having an upper bound, has a smallest upper bound, called its "least upper bound" or LUB.
This can be taken as an axiom, or with our definition of real numbers as decimals it can be proved as a theorem. I.e. the assumption that every infinite decimal does represent a real number is equivalent to the LUB or completeness property.

This then lets us prove the existence of limits in some cases. E.g. it follows that every bounded increasing sequence of real numbers has a limit, namely its least upper bound is its limit. Similarly, it follows that every non empty set of reals which is bounded below has a greatest lower bound, or GLB. These numbers are also called the "sup" and "inf" of the set. This lets us define addition and multiplication of positive reals. For instance, each infinite decimal is the limit of its finite truncations, and we can add those finite decimals. Then the sequence of sums of these finite truncations is increasing and bounded above, so has a limit and we define its limit to be the sum of the original two decimals. We proceed similarly for products. I have a detailed treatment of this construction of the real numbers as infinite decimals from an honors class I taught 20 years ago at Paideia school in Atlanta, and you are welcome to a copy of them if you like.

Using these concepts one can prove some basic facts about continuous functions. Recall a function f:[a,b]-->R, where R is the real numbers, is continuous at x0 in Dom(f) if for every e>0 there is some d>0 such that whenever x is in the domain of f and |x-x0| < d, then |f(x)-f(x0)| < e. In particular if f is continuous at x0, then f is bounded on some interval containing x0. Just take e = 1, find d, and then f is bounded on (x0-d, x0+d), since for x in that interval f(x) lies in the interval (f(x0)-e, f(x0)+e).

Theorem 1: If f is continuous everywhere on the interval [a,b], then f is bounded on the whole interval.
The statement means the set of values of f on this interval is a bounded set, i.e. there is some B such that |f(x)| ≤ B for every x in the interval.
proof: We will assume [a,b] = [0,1] for simplicity, and give the proof of an upper bound by contradiction, i.e. if f is unbounded we will find a point of [0,1] where f is not continuous. So assume f is unbounded on [0,1]. Then if we subdivide the interval into ten smaller ones, f must also be unbounded on one of these. So there is some interval, say [.4,.5] on which f is unbounded. We start the decimal expansion of the number we seek with .4. Then subdivide the interval [.4,.5] further into ten equal parts, and find an interval say [.42, .43] on which f is unbounded, and continue our expansion with .42. Continuing in this way we se how to construct recursively an infinite decimal c = .42...... in [0,1], such that f is unbounded on arbitrarily short intervals containing this number, hence on every interval containing this number. But that contradicts the fact that f is continuous at c. If we did not have infinite decimals available we could invoke the LUB axiom to find the number c as the LUB of the approximating finite decimals **QED.**

Corollary 2: If f is continuous on the closed interval [a,b] then f takes on a maximum (and minimum) value on [a,b].
proof: By the previous result f is unbounded above, and hence the set of values of f, being non empty and bounded, has a least upper bound say B. We claim B is actually a value of f. We could prove this by subdividing as above and finding a smaller interval on which B is still the LUB for f, or we can argue by contradiction as follows: If f does not take on the value B, then f(x)-B is never zero on [a,b], so the function g(x) = 1/(B-f(x)) is continuous on [a,b]. But B is the smallest upper bound of the values of f, so no smaller number than B is an upper bound. Hence for every n there are values f(xn) of f in the interval [B-1/n,B]. But if B-1/n < f(xn) < B, then (B-f(xn)) < 1/n, so 1/(B-f(xn)) > n. This shows g is unbounded on [a,b], contradicting continuity of g. This shows F assumes a maximum value. To prove there is a minimum value, just apply this result to –f. I.e. at the same point where –f has a maximum, f has a minimum **QED.**

We can also prove the intermediate value theorem from calculus.
Theorem 3: If f:[a,b]-->R is continuous, and f(a) < 0 while f(b) > 0, then for some c between a and b, f(c) = 0.
proof: We assume [a,b] = [0,1] as before for simplicity. Look at the values of f at the points .1, .2, .3,.....,.9, and find a place where f is either zero or changes from negative to positive. If f is ever found to be zero, of course we can stop, as we have our point c. if not we find a point say .3, where f(.3) <0 but f(.4) > 0. Then look at the hundredths points between .3 and .4, and find a place say .31, where f(.31) < 0 but f(.32) > 0. Continue in this way giving a recursive construction of an infinite decimal c = .31...... such that f is negative on every finite truncation of this decimal, but positive on the "roundup" of every finite truncation. E.g. if the truncation after 5 places is .31486, then f(.31486) < 0, but f(.31487) > 0. Since c is the limit of its sequence of finite truncations and also of the sequence of their roundups, by continuity f(c) is the limit of negative values of f, and also of positive values of f. the only number which is a limit of negative numbers and also of positive numbers is zero, so f(c) must be zero. **QED.**

We could also prove by the same subdivision technique that every bounded sequence of real numbers has a convergent subsequence, but we will instead do this in the plane.

These ideas extend with little change to limits of sequences and functions in the plane. The subsets we must consider are more complicated but their essential properties are still closedness and boundedness. Recall the open disc of radius r centered at a in the plane is the set {z: the distance from z to a is less than r} = {z: |z-a| < r}. Then a set is open in C if and only if it is a union of open discs, i.e. U is open if and only if for each point a in U there is some r such that all points z closer than r to a are also in U, i.e. iff for all a in U, there exists r>0 such that |z-a|<r implies z in U. Notice here that r depends on a. A set S in C is bounded iff S is contained in some disc of finite radius centered at 0, i.e. S is bounded iff there exists some r such that for all z in S, |z| < r.

Here is the key result about existence of limits in the plane.
Definition 4: A point p is called an accumulation point of a sequence {Zn} of complex numbers iff for every disc D (of positive radius) centered at p, there are infinitely many indices n such that Zn lies in D.

Theorem 5: Every bounded sequence {Zn} in the plane has an accumulation point.
proof: Since the sequence is bounded we may choose a disc of finite radius containing every point of the sequence, and thus also a finite rectangle containing them all. For simplicity assume the rectangle containing the sequence is in the first quadrant. Subdivide the rectangle into integer squares, so there are a finite number of little squares, each of length 1 on a side. Since the sequence is infinite, there is some square containing an infinite number of terms of the sequence, and suppose its lower left corner has coordinates say (13,89). Then there are an infinite number of terms of the sequence within a distance 2 of this point.

Then subdivide the sides of that square into tenths, so we have 100 smaller squares, each 1/10 on a side. Again some one at least of those small squares contains an infinite number of terms of the sequence, say with lower left corner the point with coordinates (13.9, 89.4). Then there are an infinite number of terms of the sequence within a distance 2/10 of this point. Then subdivide again further into hundredths, choose a small square containing an infinite number of elements of the sequence, say with lower left corner (13.92, 89.41). There are an infinite number of terms of the sequence within 2/100 of this point.

Continuing we have a recursive argument for the existence of a point $p = (A,B)$ whose coordinates are real numbers, such that for every positive distance $r>0$, there are an infinite number of terms of our sequence closer to p than r. Thus p is an accumulation point of our sequence. **QED.**

Definition 6: A subset S of the complex plane is called "compact" if every sequence of elements of S has an accumulation point in S.

By theorem 5, every bounded closed subset of C is compact.

Definition 7: Recall a sequence $\{Zn\}$ of complex numbers is a function Z from the positive integers to the complex plane, where the value Z at n is denoted Zn. If Z is a sequence and n(m) is a strictly increasing function from the positive integers to themselves, then the composition of n(m) followed by Z, is called a subsequence of Z. Thus the terms of $\{Zn(m)\}$ are a subset of the terms of $\{Zn\}$.

For example, the sequence 1,2,3,4,5,6....., has the subsequence 2,4,6,..........

Definition 8: A sequence $\{Zn\}$ of complex numbers converges to p iff for every $r>0$, there is a positive integer N such that $|Zn - p| < r$ whenever $n \geq N$.

Exercise 9: If p is an accumulation point of a sequence $\{Zn\}$, then there is a subsequence $\{Z(n(m))\}$ which converges to p. [Hint: let $n(1) = 1$, and then for each integer $m>1$, choose $n(m) > n(m-1)$ and such that Zn(m) is closer to p than 1/m.]

We have seen that if S is a closed and bounded subset of the plane, then every sequence in S has a subsequence which converges to a point of S.
Exercise 10: The converse is also true. [Hint: if S is unbounded choose a sequence Zn in S such that for each n, $|Zn| > n$. Then this sequence $\{Zn\}$ cannot have a convergent subsequence. If S is not closed, and p is a point of the closure but not in S, there is a sequence $\{Zn\}$ in S such that $\{Zn\} \dashrightarrow p$. Then no subsequence of $\{Zn\}$ can converge to any point other than p, hence no subsequence can converge to a point of S.]

Definition 11: A point p is an accumulation point of the infinite set S iff for every $r>0$ there are infinitely many points of S in the disc of radius r centered at p.
Theorem 12: The following four properties of a subset S of the plane are equivalent:
1) S is compact, i.e. every infinite sequence in S has an accumulation point in S,
2) Every infinite subset of S has an accumulation point in S,

3) Every infinite sequence in S has a subsequence converging to a limit in S.
4) S is closed and bounded,
proof: Exercise.

We can now prove the plane analog of theorem 1 and corollary 2.
Theorem 13: If S is any bounded closed subset of the complex plane, and f:S-->R is a continuous real valued function, then f is bounded and takes on a maximum (and a minimum) value on S.
proof: First we claim f is bounded above on S. If not, then for every n choose Zn in S with f(Zn) > n. Then by theorem 5 and exercise 8 there is a convergent subsequence {Zn(m)} of these Zn converging to a point p, which must lie in S since S is closed. But if {Zn}-->p, then {f(Zn)}-->f(p) since f is continuous at p. But this contradicts the choice of {f(Zn)} as an unbounded sequence, since a convergent sequence cannot be unbounded. I.e. if {f(Zn)}-->f(p) then eventually all terms of the sequence lie in some finite disc centered at f(p), so the sequence is bounded. This contradiction shows f is bounded above on S.

Then let B be the least upper bound of the values of f. We claim f assumes the value B. If not, then the function 1/(B-f(z)) is continuous on S. But this is impossible as in the proof of corollary 2, since f(z) comes arbitrarily near B, so the denominator gets as close as desired to zero, so the fraction is unbounded, contradicting continuity. **QED**

This lets us prove a few more very useful facts about compact sets.
Definition 14: If V is a subset of C, an "open cover" of V is a collection of open sets {Uj} such that V is contained in the union of the {Uj}, i.e. each point of V is in some Uj for some j.

Theorem 15: If V is a closed bounded subset of C, and {Uj} is an open cover of V, there is some r>0, such that for each point z of V, not only is z contained in some one Uj, but the disc of radius r centered at z is also contained in some one Uj.
proof: For each point z in V, there is some open set Uj containing z, so there is some r>0 such that the open disc of radius r centered at z is contained in that Uj. Let r>0 be chosen as large as possible, i.e. let r be the least upper bound of all r such that the disc of radius r is contained in some one Uj. Then r depends on z, so defines a function r(z) on V. It is not hard to see this function is continuous. I.e. if w is very close to z and the disc of radius r centered at z is contained in Uj, then the largest disc centered at w and contained in Uj will be almost the same as r. Then by theorem 13, the function r has a minimum on V. Since all values of r are positive it has a positive minimum. I.e. there is some constant r>0 such that for every z in V, the disc of radius r centered at z lies in some one Uj. This proves the theorem. **QED.**

Remark: The number r found in the theorem is called a "Lebesgue number" for the open covering, after the famous French mathematician Henri Lebesgue.

Now we get a very useful finiteness property for compact sets.
Theorem 16: If S is a compact subset of C, and {Uj} an open cover of S, then there is a finite subcover, i.e. some finite number of the sets Uj already cover S.

proof: Since S is bounded and closed we can find a rectangle which contains it. if r is a Lebesgue number for the open cover, and we subdivide the rectangle into small rectangles smaller than r, then for each small rectangle we can find some one open set of the cover that contains the small rectangle. But a finite number of these small rectangles cover the set V. Since each small rectangle is contained in some one open set Uj, we have a finite number of the Uj which cover the whole set V. **QED.**

The converse is also true.

Exercise 17: If a subset S in C is not compact, then some open cover of S has no finite subcover. [Hint: if S not closed, and p is a point of the closure not in S, then the complement of the closed discs of radius 1/n centered at p, gives such an open cover of S. If S is unbounded, then the open cover by disc of radius n for all n gives such a cover.]

Exercise 18: Thus the following properties are all equivalent for subsets V of C:
i) V is compact,
ii) Every sequence in V has an accumulation point in V,
iii) Every infinite subset of V has an accumulation point in V,
iv) Every sequence in V has a subsequence converging to a point in V,
v) Every open cover of V has a finite subcover,
vi) V is closed and bounded.

Theorem 16 has an important corollary for continuous functions on compact sets. Recall that if f is continuous of a set V, then given e>0, for each point p of V there is a d>0 such that |z-p| < d implies |f(z)-f(p)| < e. But d can depend on p. I.e. even though e stays the same, as you change p, you might have to choose d smaller and smaller, so there might not be any one d that works for all p in V. E.g. y = tan(x) is continuous on [0,π/2), but if we take e=1, and we choose p closer and closer to π/2, we must choose d smaller and smaller to have tan(x) varying by less than 1, in the interval (x-d,x+d). This is because tan is unbounded on every interval (x,π/2), no matter how close x is to π/2. The problem here is that the interval [0,π/2) is bounded but not closed. We prove next that when the domain of a continuous function is compact, then given e>0, a single d>0 can be chosen that will work for points in the whole domain at once.

Theorem 19: If V is a compact subset of the complex plane, and f:V-->C is continuous on V, then for every e>0, there is one d>0 such that for every point p in V, if |z-p|<d, then |f(z)-f(p)| < e.
proof: This follows almost immediately from either theorem 15 or 16. I.e. by definition of continuity, given e>0, for each q in V there is a d(q)>0, depending on q, such that |z-q|<d(q) implies |f(z)-f(q)| < e/2. Then for any two points z,w in the disc of radius d(q) centered at q, since both f(z) and f(w) differ from f(q) by less than e/2, then |f(z)-f(w)|<e.

Moreover V is covered by the open discs of radius d(q) centered at q, for all q in V. Then by theorem 15 there is a Lebesgue number d such that every disc of radius d with any center in V, lies within one of these discs. That is our d. I.e. now let p be any point of V and consider the disc centered at p of radius d. If z is any point of this disc, then both p and z lie in one of the discs of radius d(q) centered at some q in V. Hence by the argument we gave 7 lines above, the values f(p) and f(z) differ by less than e, i.e. for every p in V, and every z, if |z-p|<d, then |f(z)-f(p)| < e. **QED**

Definition 20: A function with the property in the previous theorem 19 is called "uniformly continuous" on V. I.e. f is uniformly continuous on V iff for every e>0, there exists some d>0, such that for all p and all z in V, if |z-p|<d, then |f(z)-f(p)| < e.

Exercise 21: For subsets of the Gauss sphere = CP^1 the same statements hold except in the last one we only say V is closed. Certainly finite sets are compact, and property v) lets us make some arguments for compact sets that we could make for finite sets. So in some sense compact sets generalize finite sets.

Here is one more basic property of compact sets.
Theorem 22: If V is a compact subset of the complex plane C, and f:V-->C is a continuous map, then the image set f(V) is also compact.
proof: We can use almost any version of compactness for the proof except the closed and bounded one. I.e. the image of a closed set need not be closed and the image of a bounded set need not be bounded, so it is interesting that the image of a set which is borh closed and bounded must be both closed and bounded. Let {Wn} be a sequence contained in the set of values f(V). Then there exist preimages Zn in V such that f(Zn) = Wn, for each n. Then since V is sequentially compact, there exists a subsequence {Zn(m)} converging to A in V. But then the image sequence {f(Zn(m))} converges to f(A) in f(V). Or if you like the open cover version of compactness, let {Uj} be an open cover of f(V). Then {f^(-1)(Uj)} is an open cover of V, hence there is a finite subcover f^(-1)(U1),..., f^(-1)(Uk) of V. But then every point of V maps into one of the sets U1,....,Uk, i.e. this is a finite subcover of f(V). **QED.**

Remark 23: The same argument shows that this theorem 22 also holds for maps from compact subsets of the Gauss sphere into the Gauss sphere.

There is one more important property of sets we need, that of connectedness. In the plane, connected sets are the analog of intervals in the real line.