4150fa09. Quadratic fractional transformations (rational functions of degree two)
Theorem: Let f,g be polynomials with complex coefficients, such that max(deg(f), deg(g)) = 2, and f,g have no common roots. Then:
i) (f/g) defines a surjective holomorphic map from the Gauss sphere to itself ;
ii) exactly two points have only one preimage each, all the rest have two preimages;
iii) points with only one preimage are, in some sense, the zeroes of the derivative of f/g.
proof: (**surjectivity**) If g has any roots, then these roots go to infinity, and if g has no roots, then g is constant and f has degree 2, so infinity goes to infinity. In any case infinity is in the image of f/g. If f has any roots then these roots go to 0, and if f has no roots then infinity goes to 0, so 0 is also in the image of f/g. If w is any non zero complex constant, to find preimages of w, we set f(z)/g(z) = w, and try to solve for z. multiplying out gives f(z) = wg(z), or f(z)-wg(z)=0. If g has degree less than 2, then f has degree 2 and also f-wg has degree 2, so this polynomial has solutions by the quadratic formula. If f has degree less than 2 then g has degree 2, and since w ≠ 0, this polynomial again has degree 2 and hence always has solutions. If both f, g have degree 2, with lead coefficients a, b, then as long as w ≠ a/b, the polynomial f-wg still has degree 2 and hence has solutions. If w = a/b, then infinity maps to w = a/b, and so we still have a preimage for w = a/b. Thus in all cases there are preimages of every point, so f/g is surjective.

(**number of preimages**). Since the preimages of w are the solutions of a quadratic equation, there are at most two in all cases. (That quadratic equation can never become the identically zero equation, since f and g have no common roots, so f can never equal wg. In one case, where degree(g) is less than 2, then infinity has as preimages both infinity and the roots of g, but then g has at most one root, so again there are at most two preimages.)
When are there two preimages and when is there only one?
Lemma: If we compose a QFT with a LFT, either before or after, or both, the result is again a QFT.
proof: Let the QFT be [az^2+bz+c]/[dz^2+ez+f], where either a or d is not zero, and substitute the LFT [gz+h]/[nz+m], in place of z in the QFT. Doing this cannot render either top or bottom identically zero, since that would imply one of the quadratic polynomials is zero at infinitely many points, hence was zero moriginally. After simplifying, we get then a fraction of form
[(ag^2 + bgn + cn^2)z^2 +....]/[(dg^2 + egn + fn^2)z^2+....], where neither top nor bottom is identically zero. We claim one of them has non zero coefficient of z^2. If n=0, then g≠0, so either ag^2 or dg^2 is non zero. If n≠0, and both coefficients of z^2 = 0, then a(g/n)^2 + b(g/n) + c) = 0 = (d(g/n)^2 + e(g/n) + f), and g/n is a common root of both original quadratics, contrary to hypothesis. Thus preceding a QFT by a LFT does give a QFT.
Now we compose afterwards, i.e. we substitute the QFT in place of z in the LFT [gz+h]/[nz+m], and simplify, getting:
[g(az^2+bz+c) + h(dz^2+ez+f)]/[n(az^2+bz+c) + m(dz^2+ez+f)]. If either top or bottom were identically zero, then the original quadratics would be proportional hence have two common zeroes, contrary to hypothesis. If both coefficients of z^2 were zero, in top and bottom, then the non zero vector (a,d) is in the kernel of the matrix of coefficients of the LFT, which contradicts the assumption that matrix has non zero determinant. **QED**.

Now since we know infinity always has a preimage, by composing with a LFT if necessary sending infinity to one of those preimages, we change our QFT into one with infinity going to infinity, hence with (new) g of degree less than 2. Hence we have either $[aZ^2 + bZ + c]/[Z+e]$, or $[aZ^2 + bZ + c]$. In the first case, the fraction equals w if and only if z is a solution of the quadratic equation $aZ^2 + bZ + c = w(Z+e)$, or $aZ^2 + (b-w)Z + c-e = 0$. This last equation always has at least one solution since $a \neq 0$, and it has only one solution if and only if its discriminant $(b-w)^2 - 4a(c-e) = 0$. This discriminant is itself a quadratic equation in w with lead coefficient 1, hence has at least one solution. Thus there is at least one finite point w in C that has only one preimage. By using suitable LFT's we can make this point into infinity. Now that we have infinity going to infinity and nothing else, our QFT looks like the second case $[aZ^2 + bZ + c]$. Then the finite complex number w has only one preimage precisely when $aZ^2 + bZ + c = w$ has only one solution, i.e. when the discriminant $b^2 - 4a(c-w) = 0$. This equation is linear in w with coefficient of w equal to $4a \neq 0$, hence it has precisely one solution for w. In fact the equation is $b^2 = 4a(c-w) = 4ac - 4aw$, so the unique solution is $w = [4ac-b^2]/4a = c - b^2/4a$. Thus, in addition to infinity, there is exactly one finite point which has only one preimage under the QFT $[aZ^2 + bZ + c]$. Since this form was obtained by composing our original QFT with LFT's, and those LFT's are bijections of the sphere, it follows that also the original QFT had the same behavior. I.e. every QFT is a surjection from the Gauss sphere to itself, and exactly two points of the sphere have one preimage while all other points have two preimages.

Let us check also at least in one case, that points with only one preimage are those points where the derivative of the QFT is zero. I.e. assume we have the QFT $f = aZ^2 + bZ + c$, as above. Then the derivative is $2aZ+b$ which vanishes when $Z = -b/2a$. If we plug this into f, we get $a(-b/2a)^2 + b(-b/2a) + c = b^2/4a - b^2/2a + c = c - b^2/4a$, exactly the w found above to have only one preimage. As for infinity, since we have both input and output equal to infinity, we need to calculate the derivative in the corrdinate $w = 1/z$, at $w = 0$. To change to the w coordinate for the input variable, we replace $f(z)$ by $f(1/w) = a(1/w)^2 + b(1/w) + c = [a + bw + cw^2]/w^2$, and to change to the w variable in the out put variabl;e we take the reciprocal of this getting $w^2/[a + bw + cw^2]$. Now we take the derivative with respect to the variable w, and since $a \neq 0$, we do get zero when $w = 0$. [It does NOT work to use the original polynomial in z, and set z = infinity in that derivative.]

A lot of this explicit algebra could be avoided if we knew some topology, since a 2:1 branched cover of the sphere in which every point has 2 preimages would pull back a triangulation of the sphere, i.e. a decomposition into triangles, into a triangulation with twice as many faces, vertices and edges. But then Euler's formula would fail, since we must always have $V-E+F = 2$, and doubling the number of vertcies edges and faces would make it 4. Also, if we make sure each point with only one preimage is a vertex of our triangulation, then the number of missing vertices in the pull back would equal the number of points with one preimage. Since we need $V-E+F = 2$, not 4, there are two such points. I.e. if the original triangles have V vertices, E edges, and F faces, then the pullback triangles must have $2V-2$ vertices, $2E$ edges and $2F$ faces.