4150 fa2009 Convergent sequences and series, especially power series

Review of series. Recall a series is a formal infinite sum $\sum_{n=1}^{\infty} a_n$. We asociate to it the sequence of partial sums S1 = a1, S2 = a1 + a2, S3 = a1+a2+a3,.....

and we say the series converges to a limit L if and only if the sequence of partial sums does so. Thus $\sum_{n=1}^{\infty} a_n$ = L if and only if for every e>0, there is some N such that for all n ≥ N, we have |Sn – L | = $|\sum_{j=1}^{n} a_j$-L| < e.

One necessary criterion for convergence that is easy to remember is that if a series $\sum_{n=1}^{\infty} a_n$ converges, then the individual terms go to zero, i.e. then the sequence {an}-->0. This is of course not sufficient, since the terms 1/n-->0 but the series with terms 1/n does not converge. In fact if a series converges then not just the individual terms go to zero, but the sum of all the ones after a certain point goes to zero. I.e. if $\sum_{n=1}^{\infty} a_n$ converges, then the sequence of "tail ends" $\{\sum_{j=n}^{\infty} a_j\}$-->0 as n goes to infinity. We use this fact next.

Recall the useful Cauchy criterion for convergence of sequences which gives a way to know a sequence converges without finding the limit. Recall a sequence {Sn} is Cauchy if and only if for every e>0, there is some N such that whenever n,m ≥N, then |Sn-Sm| < e. Thus in a convergent sequence with limit L, the elements of the sequence eventually get close, and stay close, to L. In a Cauchy sequence we only say they get close, and stay close, to each other. Hence they are converging to some location in the plane, and since there is always a point at every location, i.e. there are no holes in the plane, they converge.

EASY FACTS:
1) Every Cauchy sequence is bounded, i.e. if {Sn} is Cauchy, then for some R>0, we have |Sn| ≤ R for all n.
2) If a subsequence of a Cauchy sequence converges to L, then the original sequence also converges to L.

Theorem: Every Cauchy sequence converges.
proof: Since a Cauchy sequence is bounded it lies in some closed bounded disc D, and then by compactness of D, some subsequence converges in D, hence so does the original sequence. **QED.**

For series, the corresponding notion is absolute convergence. In particular this concept gives a way to conclude that a series converges without finding the limit. It is also equivalent to Cauchyness as we will partially show in the following discussion.

Recall a series $\sum_{n=1}^{\infty} a_n$ of complex numbers is called "absolutely convergent" if the corresponding series of real non negative absolute vaues $\sum_{j=1}^{n} |a_j|$ is convergent.

Theorem: An absolutely convergent series is also convergent.
proof: We will show the sequence of partial sums is Cauchy, hence convergent. This is easy. Since $\sum_{j=1}^{n} |a_j|$ converges, for every e>0 there is some N such that for all n≥N the tail ends are smaller than e, i.e. $\sum_{j=n}^{\infty} |a_j| < e$. Then for every n,m ≥ N, we have |Sn-Sm| = $|\sum_{j=n}^{m} a_j| \le \sum_{j=n}^{m} |a_j| \le \sum_{j=n}^{\infty} |a_j| < e$. I.e. the sequence {Sn} of partial sums is Cauchy. **QED.**

Recall an absolutely convergent series can be reordered in any order and it will still converge to the same limit. One nice thing about power series is they will always be absolutely convergent everywhere inside their disc of convergence.

The most fundamental series of all is the geometric series. It is essential to be closely familiar with this series, since it is the basis for comparison for all other series we will consider. Indeed power series are just modifications of geometric series, and this is what gives them their nice behavior.

Geometric series
If r is a number, the series a + ar + ar^2 + ar^3 +...... = a(1 + r + r^2 + r^3 +....) is called the geometric series with ratio r. These series may or may not converge. E.g. the series 1 + 1 + 1^2 + 1^3 +... with ratio 1 does not converge, nor does 1 + 2 + 2^2 + 2^3 +.... with ratio 2.

However if 0 < |r| < 1, then the series does converge. I.e. for all n, we have
1 + r + r^2 +....+r^n = [1- r^(n+1)]/(1-r). (**exercise**)
Thus if |r|<1, then r^n-->0, as we have seen, so the right hand side approaches 1/[1-r]. The point here is that there is a number r < 1, such that each term is obtained by multiplying the previous one by r.

We need one more easy test for convergence that does not force us to find the limit. Recall every bounded increasing sequence converges. This leads to the following:

Comparison test
Let $\sum_{n=1}^{\infty} a_n$ be a sequence of non negative terms that converges. Then the sequence of partial sums is (weakly) increasing, hence it converges if and only if it is bounded. Thus if $\sum_{n=1}^{\infty} a_n$ converges, and $\sum_{n=1}^{\infty} b_n$ is another non negative series such that eventually bn ≤ an,

for all large n, then the series $\sum_{n=1}^{\infty} b_n$ also converges. In particular if z is a number and {an} is a sequence such that for all large n we have |anz| < 1, then the series $\sum_{n=1}^{\infty} (a_n z)^n$ converges absolutely by comparison with the geometric series. Or equivalently, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if |(an)^(1/n) z| < 1 for all large n.

This gives us these tests:
Root test: If for all large n, |an^(1/n)| ≤ R, or if |an^(1/n)| -->R, and |z| < 1/R, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. In particular, if |an^(1/n)|-->0, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z.

proof: In either case, for all large n, we have |(an)^(1/n) z| < 1, as noted above. QED.

Ratio test: If the ratios |an+1/an|-->R, or if even |an+1/an| ≤ R for all large n, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for |z| < 1/R. In particular if |an+1/an|-->0, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z.
proof: If |an+1/an| ≤ R, for all large n, and |z|< 1/R, then after a certain point each term equals the previous one multiplied by |(an+1/an) z| which is less than R|z| = r < 1, hence we may compare to a geometric series with ratio r again. The limit case is similar. **QED.**

A more careful, similar analysis yields this result:
If R is the largest number such that some subsequence of |an^(1/n)| converges to R, then the series $\sum_{n=1}^{\infty} a_n z^n$ converges absolutely for |z|} < 1/R, but diverges for |z| > 1/R. (compare Lang p.53-56, section II.2.) If R = limsup{|an^(1/n)| }, then r = 1/R is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

We want to know that convergent power series are continuous and differentiable within the disc of convergence. These proofs use the concept of "uniform" convergence. For this it is convenient to define a sort of absolute value for functions, at least for bounded functions.

Definition: If f is a bounded function on a set S, we define the "supnorm" of f on S to be the lub of all values of |f| on S, denoted $\|f\|_S$ or just ||f||, if the set S is understood.

Then we say a sequence of bounded functions {fn} on S converges to the function g in the sup norm, or uniformly, if for all e>0, there is some N such that for all n≥N, we have ||fn-g|| < e. I.e. all functions fn for large enough n, differ at every point z of S, by less than e, from the value g(z).

The Cauchy criterion and comparison tests hold for this type of convergence also, and we have: "Weierstrass M-test"

If $\sum_{n=1}^{\infty} a_n$ is a convergent series of non negative terms (Weierstrass used M's instead of a's), and if for all large n, we have $\|f_n\| \leq a_n$, then the series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly on S to a function f.

In particular for every z in S, the series of values $\sum_{n=1}^{\infty} f_n(z)$ converges to $f(z)$.

Uniform convergence is especially well behaved in the following sense.

Theorem: If f is the uniform limit of continuous (bounded) functions on S, then f is also continuous (and bounded) on S.

Corollary: If f is a function which on every closed disc D is a uniform limit of continuous functions, then f is continuous everywhere in the plane. Or if f is defined only in the open disc Dr, and is the uniform limit of continuous functions on every smaller closed disc Ds for s< r, then f is continuous on Dr.

This helps us deal with power series then because of the next result.

Theorem: If $\sum_{n=0}^{\infty} a_n z^n$ is a power series convergent in the open disc of radius r, then for every s<r, the series is uniformly (in particular absolutely) convergent on the disc Ds. hence the series defines a continuous function in all of Dr.

Even uniform convergence does not always guarantee differentiability of the limit function, but fortunately for power series it does.

Theorem: If f(z) = $\sum_{n=0}^{\infty} a_n z^n$ is a complex power series convergent in the open disc of radius r, then the limit function is infinitely differentiable in Dr, and for each n, the nth "derived series" converges in Dr to the nth derivative, i.e. you can differentiate term by term as many times as you like.

We will not prove this but the proof is in Greenleaf, 3.3, p.126-130, and Lang, II.5, p.72.

Analytic functions

Our main goal is to introduce a class of holomorphic functions more general than the polynomials and rational functions we have focused on so far. For example, we want to include e^z and cos(z) which are not polynomials, because they are not finite to one or not surjective. E.g. cos(z) has an infinite number of preimages of 0, namely z = π/2 + nπ for all integers n, and e^z has no zeroes at all.

Definition: A function f is <u>analytic</u> in an open set U if for every point c in U, there is an open disc D centered at c, and a power series centered at c and convergent to f(z) for all z in D.

Thus for all z near c, f(z) can be represented as
a0 + a1 (Z-c) + a2 (Z-c)^2 +....+an(Z-c)^n +...... Since this power series will in general converge only in a disc, in case the set U is not a disc, then we will have to use different power series at different points of U. In some cases, such as f(z) = e^z, the disc can be taken as the whole plane and we only need the one series 1 + Z + Z^2/2! + Z^3/3! +..... However even in this case it may be convenient at some times to re – expand e^z as a power series centered at some other point c.

Then about the point c, it turns out we get the series
e^Z = e^c + e^c (Z-c) + e^c (Z-c)^2/2! + e^c (Z-c)^3/3!+.......
This reflects the fact that e^Z = e^c.e^(Z-c) for all c and all Z.

In fact by definition of analytic, to prove e^z is analytic we need to show it can be expressed as a series centered at every point, not just at 0, so the possibility of such a re – expansion is part of the definition of analytic function.

This re - expansion technique is related to the root factor theorem for polynomials. I.e. if we want to display a polynomial in the form f(z) = f(c) + (z-c)^kg(z) where g(c) ≠ 0, instead of using the root factor theorem, we could just re expand f(z) as f(z-c+c) using the binomial theorem. Each term Z^k becomes ((Z-c)+c)^k =
(Z-c)^k + kc(Z-c)^(k-1)+(k)(k-1)/2 c^2 (Z-c)^(k-2)+...+kc^(k-1)(Z-c) + c^k.
Thus combining all these terms obtained from re expanding a0 + a1Z + a2Z^2 +...anZ^n, we get some epression like b0 + b1(Z-c) + b2(Z-c)^2 +...+bn(Z-c)^n. Now if bk is the first non zero coefficient after b0, we get the expression

b0 + (Z-c)^k g(Z), where g(Z) = bk + bk+1 (Z-c) +....+bn(Z-c)^n, and where bk = g(c) ≠ 0. This is what we had from the root factor theorem before.

We can do the same thing within the disc of convergence for a power series. I.e. given an infinite power series a0 + a1 Z + a2 Z^2 +....+anZ^n +...... we can replace each Z by (Z-c)+c, and re expand each term akZ^k = ak((Z-c)+c)^k as above, to get
ak[(Z-c)^k + kc(Z-c)^(k-1)+(k)(k-1)/2 c^2 (Z-c)^(k-2)+...+kc^(k-1)(Z-c) + c^k]. We have an infinite number of these expressions, one for each k, and we add them all up, rearranging according to the power of (Z-c). The power (Z-c)^n occurs once in each of these expressions for which k ≥ n, and so for each n we have an infinite sum of terms as the coefficient of (Z-c)^n.

If you stare at those terms you can see that they are the terms for the nth “derived series” expansion for the nth derivative of f, evaluated at c. Indeed that follows from the theorem that we can differentiate a convergent power series term by term. Thus the coefficient of the power (z-c)^n must be f^(n)(c)/n!

We should also practice some algebra with power series such as multiplying and dividing them, and substituting one into another, being careful only to substitute g into f if the constant term of g is zero. These operations are related, e.g. since divisino by $(1-z)$ gives $1/(1-z) = 1+z + z^2 + z^3 + \ldots.$, so also $1/[1-\sin(z)] = 1 + \sin(z) + \sin^2(z) + \sin^3(z) + \ldots..$, since the constant term of $\sin(z)$ is $\sin(0) = 0$, and where on the right we substitute for $\sin(z)$ everywhere its power series. Lang has an example on p.41 and some practice exercises on p.46.

E.g. $1/[1-\sin(z)] = 1 + [z - z^3/3! + \ldots] + [z - z^3/3! + \ldots]^2 + \ldots\ldots = 1 + z + z^2 - z^3/36 - z^4/3 + z^5/120 + z^6/36 \pm \ldots\ldots$