

4150 fall 2009, lecture notes 10-5-09, Fundamental theorem of algebra, part one

The fundamental theorem of algebra usually says that a non constant polynomial with complex coefficients has at least one complex root, i.e. if $f(z) = a_0 + a_1 z + \dots + a_n z^n$, is a polynomial of degree $n \geq 1$, there is some complex number c such that $f(c) = 0$. Thus the equation $f(z) = 0$ always has a solution for z . But this implies every equation of form $f(z) = a$, also has a solution, since we could apply the original statement to find a solution of the polynomial equation $f(z) - a = 0$. I.e. if $f(z)$ is a non constant polynomial, so is $f(z) - a$.

So we will prove the following theorem: every non constant complex polynomial $f(z)$ defines a surjective map $f: \mathbb{C} \rightarrow \mathbb{C}$, where \mathbb{C} denotes the set of complex numbers. Our technique however will be to use compactness, and for that we need to enlarge the set \mathbb{C} to the Gauss sphere \mathbb{CP}^1 . So what we will actually prove is this:

Theorem: If $f(z)$ is any polynomial of degree $n \geq 1$ with complex coefficients, and if we extend f to $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, by setting $F(\infty) = \infty$, we obtain a surjective map from the Gauss sphere to itself, such that the only point mapping to infinity, is infinity. In particular the restriction of F to \mathbb{C} is f , which is thus also surjective.

proof:

Part one: Every continuous, open map $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is surjective.

Part two: Every non constant polynomial map $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, is a continuous open map. This part will be delayed.

Our goal today is to see how to prove part one using the concepts we have been building up of compact, closed, and open sets. There are several steps.

I. Claim: \mathbb{CP}^1 is compact.

proof: If $\{z_n\}$ is any sequence in \mathbb{CP}^1 we want to prove there is a subsequence converging to a point of \mathbb{CP}^1 . We know this for sequences contained in bounded closed subsets of \mathbb{C} , so we are done if there is some closed disc $D = \{z: |z| \leq r\}$ which contains an infinite number of elements of our sequence, since then that subsequence has itself a convergent subsequence with limit in D . So we may assume for each m , that the finite disc $D_m = \{z: |z| \leq m\}$ contains only a finite number of elements of our sequence. Thus for each m , we may choose an element $z_{n(m)}$ of our sequence such that $n(m) > n(m-1)$ and such that $|z_{n(m)}| \geq m$. Then this subsequence converges to infinity by definition of convergence to infinity. [Recall a sequence converges to infinity iff for every $r > 0$, eventually all elements of the sequence are outside the disc $D = \{z: |z| \leq r\}$. By our choice of subsequence, all we have to do is choose $m > r$, and we have that all the elements $z_{n(k)}$ with $k > m$ are outside that disc.]

Thus in all cases our sequence has a convergent subsequence, so \mathbb{CP}^1 is sequentially compact. **QED.**

II. If $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is continuous, and $L = F(\mathbb{CP}^1)$ is the image of F , then L is also compact. (**I.e. the continuous image of a compact set is compact.**)

proof: If $\{w_n\}$ be a sequence in L , we want to find a subsequence converging to a point of L . By definition of L , every point w_n is the image of a point of \mathbb{CP}^1 , i.e. for each n , there is some z_n in \mathbb{CP}^1 with $F(z_n) = w_n$. Then by compactness of \mathbb{CP}^1 , the sequence

$\{z_n\}$ has a subsequence $\{z_{n(m)}\}$ converging to some point c in $\mathbb{C}P^1$. Then by continuity of F , the image sequence $\{w_{n(m)}\} = \{F(z_{n(m)})\} \rightarrow F(c)$. Thus the subsequence $\{w_{n(m)}\}$ converges to the point $F(c)$ which is also in $L = F(\mathbb{C}P^1)$, since $F(c)$ is the image of the point c . **QED.**

III. The set $L = F(\mathbb{C}P^1)$ above is also closed, (i.e. **every compact set in $\mathbb{C}P^1$ is closed**). (For subsets of \mathbb{C} we know all compact sets are closed and bounded, but in $\mathbb{C}P^1$ some of them are technically unbounded, at least in the absolute value sense for z , such as $\mathbb{C}P^1$ itself, and we need to prove that in $\mathbb{C}P^1$ all compact sets are closed. This was left as exercise 21 on page 7 of my hand out notes on compact sets. See also theorem 22 of those notes for a different proof of step II above, and remark 23.)

proof: If c is a boundary point of L , we claim c is already in L . Let $\{z_n\}$ be a sequence in L converging to c in $\mathbb{C}P^1$. We want to prove that c is also in L . By the compactness of L , there is a subsequence $\{z_{n(m)}\}$ converging to a point d in L . By math 3100, this subsequence has the same limit as the original sequence, so $d = c$, hence c is in L . **QED.**

Exercise: Every closed subset of $\mathbb{C}P^1$ is also compact. [This should be easy.]

Corollary: If $F: \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ is defined by a continuous, open map, then the image set $L = F(\mathbb{C}P^1)$, is both open and closed in $\mathbb{C}P^1$, (and non empty).

proof: Since the map is open, i.e. takes open sets to open sets, and since $\mathbb{C}P^1$ is itself open, L is open. Our discussion in part two above shows the image set L is also closed. L is certainly non empty since it contains for example the point $F(0)$. **QED.**

The next result will thus finish off part two:

IV. Theorem: The only non empty set in $\mathbb{C}P^1$ which is both open and closed is $\mathbb{C}P^1$ itself. (The essential point here is that $\mathbb{C}P^1$ is "connected", as we will discuss later.)

proof:

Lemma: If K were a non empty subset of $\mathbb{C}P^1$ which is both open and closed but not equal to all of $\mathbb{C}P^1$, then the complement $L = (\mathbb{C}P^1 - K)$ would also be non empty, and open and closed.

proof: We only need to check that the complement of an open set is closed, and vice versa. By the symmetry of the definition of boundary points, a set and its complement have the same boundary points. Since an open set contains none of its boundary points, it follows that the complement contains all of them, hence contains all its own boundary points, hence is closed. In the same way if a set contains all its boundary points, then the complement contains none of them hence none of its own boundary points, hence is open. **QED.**

Thus if $\mathbb{C}P^1$ had a non empty subset K which is both open and closed, and not equal to all of $\mathbb{C}P^1$, then we could write $\mathbb{C}P^1$ as a union of two disjoint non empty sets K, L such that K and L are both open, (and both closed). Intuitively this means $\mathbb{C}P^1$ could be disconnected into two pieces, which we will show leads to a contradiction.

Lemma A: Assume K, L are non empty disjoint subsets of \mathbb{CP}^1 , both open, which together cover all of \mathbb{CP}^1 . Then there is a continuous function $f: \mathbb{CP}^1 \rightarrow \mathbb{R}$, with $f(z) = -1$ whenever z is in K , and $f(z) = 1$ whenever z is in L .

proof: If p is a point of K , then since K is open, no sequence of points from outside K can converge to p . Thus if $\{z_n\}$ is a sequence converging to p , the sequence eventually gets entirely inside K , so the sequence of values $\{f(z_n)\}$ is eventually constant, equal to -1 , hence $\{f(z_n)\}$ converges to $-1 = f(p)$. Similarly if q is in L , and $\{z_n\}$ is a sequence converging to q , then $\{f(z_n)\} \rightarrow 1 = f(q)$. Thus f is continuous. **QED.**

Now we have a continuous function on \mathbb{CP}^1 whose only values are -1 and 1 . We are almost ready to get a contradiction to the intermediate value theorem from calculus. We just need a continuous function defined on an interval in \mathbb{R} , with values only -1 and 1 .

Lemma B: If p and q are any two points of \mathbb{CP}^1 , there is a continuous function $g: [0,1] \rightarrow \mathbb{CP}^1$, such that $g(0) = p$ and $g(1) = q$.

proof: If p, q are both complex numbers, just take $g(t) = tq + (1-t)p$. If q is infinity, and $p \neq 0$, let $g(t) = p/(1-t)$, for $0 \leq t < 1$, and $g(1) = \text{infinity}$. If $p = 0$, and $q = \text{infinity}$, define $g(t) = t/(1-t)$. **QED.**

[Lemma B says any two points on the sphere can be connected by a continuous path.]

proof of part one:

Now if we combine our results from Lemmas A and B, we see that IF there were a non empty subset K of \mathbb{CP}^1 which were both open and closed, yet not equal to all of \mathbb{CP}^1 , then the composition $(f \circ g): [0,1] \rightarrow \mathbb{R}$, would be a continuous function with $f(g(0)) = -1$, and $f(g(1)) = 1$, and yet the function never takes the value zero anywhere on the interval $[0,1]$. This contradicts the intermediate value theorem, and proves that if $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is any continuous open map, then the non empty, open and closed subset, $F(\mathbb{CP}^1)$ is in fact all of \mathbb{CP}^1 , i.e. F is surjective. This completes the proof of part one. **QED.**

Next task: It remains for us to show that if f is any non constant complex polynomial, then f defines a continuous, open, map $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ (with only infinity mapping to infinity), hence both F , and the original $f: \mathbb{C} \rightarrow \mathbb{C}$ are surjective. Our method will be to invoke the "inverse function theorem" from real calculus (which we will recall without proof), to show that every non constant polynomial behaves everywhere locally like some z^k , with $k \geq 1$, hence is open. So it all boils down to the fact that for complex functions the power maps z^k are open for $k \geq 1$. Note this fails for real power maps: i.e. x^2 for instance is not an open map on the interval $[-1,1]$, since it takes the interior point 0 of the domain, to the boundary point 0 of the image interval $[0,1]$. Please read Greenleaf pages 99-103 to review inverse functions, especially theorem 2.21, the "open mapping theorem" on page 103.

NOTE: however, we are going to prove a stronger open mapping theorem than 2.21, since we will not assume the derivative of the holomorphic map is never zero, only that it is not always zero. This theorem will take us a long time to prove for holomorphic functions, but we will prove it soon for polynomials.

