

4150 fa09. Fundamental theorem of algebra, part 2

To finish up we want to prove that every non constant complex polynomial defines a continuous, open map $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, where $F(z) = f(z)$ if z is a complex number, and $F(\infty) = \infty$.

A. We know a polynomial is differentiable, and a differentiable function is continuous as a function from \mathbb{C} to \mathbb{C} . So we only need to prove that F is also differentiable at infinity. Since $F(\infty) = \infty$, and since when $z = \infty$, we use the coordinate $w = 1/z$, hence $z = 1/w$ at $w = 0$, this means showing that the function $1/F(1/w) = 1/f(1/w)$ is differentiable at $w = 0$. I.e. since both input and output equal infinity, we must use the reciprocal in both domain and range.

So if $f(z) = a_0 + a_1 Z + a_2 Z^2 + \dots + a_n Z^n$, where $a_n \neq 0$, then $f(1/W) = a_0 + a_1/W + a_2/W^2 + \dots + a_n/W^n = [a_1 W^{(n-1)} + a_2 W^{(n-2)} + \dots + a_n] / W^n$, so $1/f(1/W) = W^n / [a_1 W^{(n-1)} + a_2 W^{(n-2)} + \dots + a_n]$. To show differentiability at $Z = \infty$, means showing differentiability of this function at $W = 0$. Notice that at $W = 0$, the denominator equals $a_n \neq 0$, so this fraction is differentiable at $W = 0$, by the quotient rule, hence also continuous. Thus $F: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is continuous everywhere. **QED.**

B. Now to show F is open.

I. F is open at a point p in \mathbb{C} where $f'(p) \neq 0$.

proof: This follows from the real open mapping theorem as in Greenleaf, page 103. I.e. when $f(z) = u(z) + iv(z) = u(x+iy) + i v(x+iy)$, the matrix of real partials of (u,v) has rows $[\partial u/\partial x \ \partial u/\partial y] \ a, [\partial v/\partial x \ \partial v/\partial y]$, with determinant $D = (\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) = (\partial u/\partial x)^2 + (\partial v/\partial x)^2$, by the Cauchy Riemann equations. Hence if $f'(p) \neq 0$, this determinant is non zero, so the matrix of partials is invertible, and the linear map it defines, i.e. the differential of f at p , is an invertible linear map. Since f is a polynomial these partials are also polynomials hence continuous, so the real mapping defined by f is continuously differentiable with invertible differential at p . Under these hypotheses the real inverse mapping theorem says there are open sets U, V with p in U and $f(p)$ in V , such that f defines a smooth isomorphism $f: U \rightarrow V$. I.e. f is a bijection from U to V , and the inverse map $g: V \rightarrow U$ [such that for z in U and w in V , we have $g(w) = z$ if and only if $f(z) = w$], is also continuously differentiable.

In particular, $f(p)$ is an interior point of the image set $V = f(U)$, hence also of $F(\mathbb{CP}^1)$.

The same argument works at infinity in the coordinate $w = 1/z$, provided the derivative of $1/f(1/w)$ with respect to w is non zero, which is true in our case if and only if $n = 1$. I.e. $1/f(1/W) = W^n / [a_1 W^{(n-1)} + a_2 W^{(n-2)} + \dots + a_n]$ has derivative zero at $W = 0$, by the quotient rule, whenever $n \geq 2$.

II. If $k \geq 1$, the map Z^k is open, on every disc centered at 0.

proof: If D is an open disc centered at 0, we want to show for every point Z in D , that Z^k is an interior point of the image of D . Since the derivative of Z^k only vanishes at 0, this follows from the previous case at all points except $Z = 0$. So we must show that 0 is an interior point of the image of D . But we know from the geometry of multiplication, that the k th power map takes the disc of radius r centered at 0, surjectively onto the disc

of radius r^k centered at 0. I.e. every complex number of length less than r^k , has a k th root of length less than r .

III. Now we want to deal with points where then derivative equals zero. E.g. at $z = \infty$, we have $1/f(1/W) = W^n / [a_1 W^{(n-1)} + a_2 W^{(n-2)} + \dots + a_n]$, which has derivative zero at $W = 0$, by the quotient rule, whenever $n \geq 2$. But here we can factor out W^n , getting $W^n \cdot [1/[a_1 W^{(n-1)} + a_2 W^{(n-2)} + \dots + a_n]]$, where the second factor equals $1/[a_n] \neq 0$, at $W=0$. This is typical of points where the derivative equals zero. In fact here the first $n-1$ derivatives vanish at $w = 0$.

If p is any complex number, we can always factor $f(z)$ as $(z-p) \cdot h(z) + f(p)$, by long division, i.e. by the "remainder theorem" from high school. Hence $f(p) = 0$ if and only if $(z-p)$ divides $f(z)$. Then $f'(z) = h(z) + (z-p) \cdot h'(z)$, so $f'(p) = 0$ if and only if $h(p) = 0$, if and only if $(z-p)$ divides $h(z)$, iff $f(z) = (z-p)^2 \cdot h_1(z) + f(p)$. Continuing, we conclude the first $k-1$ derivatives of f vanish at p , but not the k th, if and only if we can write $f(z) = (z-p)^k \cdot h(z) + f(p)$, where $h(p) \neq 0$.

Now let $f(z)$ be a complex holomorphic function of form $f(z) = (z-p)^k \cdot h(z) + c$, where h is holomorphic and $h(p) \neq 0$. Then we claim f is open at $z = p$.

To prove it we will write f as a composition of holomorphic functions near p . Since $h(p) \neq 0$, we claim we can choose a holomorphic k th root of h near $h(p)$. I.e. since log can be defined holomorphically near any non zero complex number, we can find a holomorphic function $g(z) = e^{\{(1/k)(\log(h(z)))\}}$, defined on an open set containing p , and such that $[g(z)]^k = h(z)$ for z in that open set.

Then $f(z) = (z-p)^k \cdot [g(z)]^k + f(p) = [(z-p) \cdot g(z)]^k + f(p)$, in this open set. Hence on this open set $f(z) - f(p) = [(z-p) \cdot g(z)]^k$, is a composition of the function $[(z-p) \cdot g(z)]$, with the k th power function $[]^k$. We claim these are both open. We already know the power function $[]^k$ is open so consider $[(z-p) \cdot g(z)]$. Since $g^k(p) = h(p) \neq 0$, hence also $g(p) \neq 0$. Thus the derivative $[(z-p) \cdot g(z)]' = g(z) + (z-p) \cdot g'(z)$, equals $g(p) \neq 0$ at $z=p$, hence this map is open near p by the real inverse function theorem above in step I.

We have shown that near every point, even infinity (in w - coordinates), a complex polynomial can be written as $f(z) = (z-p)^k \cdot h(z) + f(p)$ for some $k \geq 1$, where h is holomorphic and $h(p) \neq 0$, and that such a function is open near p . Hence a complex polynomial does define an open mapping on \mathbb{CP}^1 .

Remark: We have not used this fact above, but the chain rule says that the differential of the inverse function in part I, is the matrix inverse of the differential of the original function. Since the inverse of a matrix satisfying the Cauchy Riemann equations also satisfies them, it follows that the inverse function of a continuously holomorphic function is also continuously holomorphic. We did not need this in our proof since we only used the holomorphicity of the inverse function for log, where we knew it already.