

4150 fall 2009 Open mapping principle for holomorphic functions

Holomorphic isomorphisms

Assume U, V are open sets in the complex plane \mathbb{C} and $f: U \rightarrow V$ is a holomorphic map. We say f is a holomorphic isomorphism from U to V , iff there is a holomorphic map $g: V \rightarrow U$ such that for all z in U $g(f(z)) = z$, and for all z in V $f(g(z)) = z$. then both f, g are bijections, and we call g the (composition) inverse of f . Obviously then g is also a holomorphic isomorphism from V to U , and f is the inverse of g .

Examples: i) The map $f(z) = e^z$ is an isomorphism from the set $U = \{z: 0 < \text{Im}(z) < \pi\}$ to the set $V = \{z: \text{Im}(z) > 0\}$, with inverse $g(z) = \text{Log}(z) = \ln(|z|) + \text{Arg}(z)$ where we choose $0 < \text{Arg}(z) < \pi$ for z in the upper half plane.

ii) The map $f(z) = z^2$ is an isomorphism from the open set $U = \{z: \text{Re}(z) > 0, \text{Im}(z) > 0\}$ to the open set $V = \{z: \text{Im}(z) > 0\}$, with inverse $g(z) = \sqrt{z} = e^{[(1/2)(\text{Log}(z))]}$, where again we choose $\text{Log}(z) = \ln(|z|) + i \text{Arg}(z)$, with $0 < \text{Arg}(z) < \pi$, for z in V .

Definition: If U is open in \mathbb{C} , a map $f: U \rightarrow \mathbb{C}$ is called an “open map” iff for every open subset W in U , $f(W)$ is open in \mathbb{C} .

Exercise: f is open on U iff for every p in U , there is some smaller open set W in U on which f is open, iff for every p in U and all sufficiently small $r > 0$, $f(p)$ is an interior point of the image $f(D_r(p))$ of the open disc of radius r centered at p .

Remark: If f is open on some open set containing p , we say f is “locally open” at p , hence by the exercise, f is open on U iff it is locally open at every point of U .

Lemma: If U, V are open and $f: U \rightarrow V$ is a bijection, with inverse function $g: V \rightarrow U$, then f is open iff g is continuous.

proof: For every open W in U , $f(W) = g^{-1}(W)$, so f is open iff the images of open sets are open under f , iff their inverse images are open under g iff g is continuous. **QED.**

Corollary: If $f: U \rightarrow V$ is a holomorphic isomorphism with inverse $g: V \rightarrow U$ then both f, g are continuous hence both are open mappings.

Exercise: A composition of open mappings is open.

Local isomorphisms

If $f: U \rightarrow \mathbb{C}$ is holomorphic, we say f is a “local isomorphism at p ” if there is an open subset W of U containing p , such that $f(W) = V$ is open in \mathbb{C} and the restriction $f: W \rightarrow V$ is a holomorphic isomorphism.

Inverse function theorem: If $f: U \rightarrow \mathbb{C}$ is holomorphic and if p in U is a point where $f'(p) \neq 0$, then f is a local isomorphism at p . In particular, there is some open set W containing p such that the restriction of f to W is an open mapping.

Corollary: If $f'(p) \neq 0$ for all p in U , then $f: U \rightarrow \mathbb{C}$ is an open mapping.

proof: It is everywhere locally open, hence open. **QED.**

Example: $f(z) = e^z$ is an open mapping $\mathbb{C} \rightarrow \mathbb{C}$ since the derivative is never zero. Note however that f is not an injection.

However this hypothesis is much too strong, as in fact every non constant holomorphic map $\mathbb{C} \rightarrow \mathbb{C}$ is open. We will obtain next a partial result in this direction, for polynomials and rational functions.

Basic case of open maps with derivative zero

The map $f(z) = z^n$ is open, for all $n \geq 1$, even at $z=0$, where $f'(0) = 0$.

proof: At every non zero point, f is a local isomorphism hence open, so we only need to look at $z=0$. It suffices to show that 0 is an interior point of $f(D_r(0))$ for all small $r > 0$. But $f(D_r(0)) = D_{r^n}(0)$, and 0 is the center of this disc, hence certainly an interior point.

QED.

Note more simply, the boundary of the disc $D_r(0)$ maps to the boundary of $D_{r^n}(0)$ and every interior point z with $|z| < r$, maps to an interior point z^n with $|z^n| < r^n$. Similarly, $(z-a)^n$ is an open map near $z=a$, as is $(z-a)^n + b$.

Second case of an open map with zero derivative.

If $f(z) = (z-a)^n g(z) + b$, where g is holomorphic and $g(a) \neq 0$, then f is open near $z=a$.

proof: Choose $h(z)$ such that for z near a , h is holomorphic and $h^n(z) = g(z)$. Since $g(a) \neq 0$, for z near a we can choose a holomorphic branch of Arg near $g(a)$, and define $h(z) = e^{[(1/n)(\ln(|g(z)|) + i\text{Arg}(g(z)))]}$. Then $f(z) = [(z-a)h(z)]^n + b$. And the derivative of the inner function $(z-a)h(z)$ at $z=a$, equals $h(a) \neq 0$, so on some open set around a , $(z-a)h(z)$ is an isomorphism hence open. Since $f(z)$ is the composition of the local isomorphism $(z-a)h(z)$, with the n th power function which is open, f is also locally open near a . **QED.**

I.e. if we think of $w = (z-a)h(z)$ as a local coordinate near a , then in this w -coordinate system $f(z(w)) = w^n + b$, which is open by the basic case.

Corollary: If $f(z) = a_0 + a_1 z + \dots + a_n z^n$ is a polynomial of degree $n \geq 1$, i.e. $a_n \neq 0$, then f defines an open map $\mathbb{C}^1 \rightarrow \mathbb{C}^1$, where $f(\infty) = \infty$.

proof: If c is a finite point then $z = c$ is a root of $f(z) - f(c)$, so $(z-c)$ divides $f(z) - f(c)$ by the division theorem from high school algebra. If we divide $(z-c)$ out as many times as possible from $f(z) - f(c)$, we get $f(z) - f(c) = (z-c)^k g(z)$ for some $n \geq k \geq 1$, where g is a polynomial of degree $n-k$ and $g(c) \neq 0$. (Taking derivatives by the product rule shows that the first $k-1$ derivatives of f vanish at $z=c$ but not the k th.)

By the previous lemma, f is locally open at $z=c$. Note that f is a local isomorphism near c iff $k = 1$, so this argument covers all cases.

Now consider the point $z = \infty$. Here we have both in-out and output equal to $z = \infty$, so we must calculate in the local coordinate $w = 1/z$, or $z = 1/w$. Thus we consider the function $1/(f(1/w))$ near $w = 0$. Expanding gives $1/[a_0 + a_1/W + \dots + a_n/W^n] = W^n/[a_0 W^n + a_1 W^{(n-1)} + \dots + a_{(n-1)}W + a_n]$, where $a_n \neq 0$. Thus this has the form in the lemma, of $W^n (1/[a_0 W^n + a_1 W^{(n-1)} + \dots + a_{(n-1)}W + a_n]) = W^n g(W)$ where $g(0) = a_n \neq 0$. Hence this function is locally open near $W=0$, i.e. near $Z=\infty$. **QED.**