4.14 Carry out the first three iterations of the solution of the following system of equations using the Gauss–Seidel iterative method. For the first guess of the solution, take the value of all the unknowns to be zero.

$$8x_1 + 2x_2 + 3x_3 = 51$$
$$2x_1 + 5x_2 + x_3 = 23$$
$$-3x_1 + x_2 + 6x_3 = 20$$

Solution

The essence of the Gauss-Seidel iterative method is given by Eq. (4.51):

$$x_i = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{j = 1 \\ j \neq i}}^{j = n} a_{ij} x_j\right)\right] \qquad i = 1, 2, \ldots, n$$

First Iteration:

Starting with $\left[x_1^{(0)}\ x_2^{(0)}\ x_3^{(0)}\right] = \left[0\ 0\ 0\right]$, $x_1^{(1)} = \dfrac{51-2(0)-3(0)}{8} = 6.375$, $x_2^{(1)} = \dfrac{23-2(0)-0}{5} = 4.6$, and $x_3^{(1)} = \dfrac{20+3(0)-0}{6} = 3.3333$.

Second Iteration:

$$x_1^{(2)} = \frac{51-2(4.6)-3(3.3333)}{8} = 3.9750,\ x_2^{(2)} = \frac{23-2(6.375)-3.3333}{5} = 6.9167,\ \text{and}$$

$$x_3^{(1)} = \frac{20+3(6.375)-4.6}{6} = 5.7542.$$

Third Iteration:

$$x_1^{(3)} = \frac{51-2(6.9167)-3(5.7542)}{8} = 2.488,\ x_2^{(2)} = \frac{23-2(3.9750)-5.7542}{5} = 1.8592,\ \text{and}$$

$$x_3^{(1)} = \frac{20+3(3.9750)-6.9167}{6} = 4.1681.$$