MATH 522/722 Class notes

Classifying all isometries of the Euclidean plane

Our first big theorem established that any triangle could be mapped onto any other congruent triangle by at most a sequence of three reflections. Thus since all isometries take a triangle to some congruent triangle, we can always find a composition of three reflections that agree on a given triangle with any given isometry. Since two isometries that agree on a triangle are equal everywhere, we get the corollary that every isometry equals a composition of at most three reflections. Consequently to classify all isometries it suffices to consider products of ≤ 3 reflections, most cases of which are easy.

Case zero: A product of zero reflections is the identity.
Case one: A product of one reflection is a reflection.
Case two: A product of two reflections is either a translation or a rotation (incl. a half - turn.)
Case three: for this we need the following:

Theorem: A product of three reflections is either a reflection or a glide.

The proof of this last theorem was fairly unintelligible (at least unmotivated, and not completely detailed) in the book, so the class gave an independent proof as follows, inspired by the following idea of Denise Spangler, a student in the course:

Proposition 1: An isometry © with precisely one invariant line is a glide.
Proof:
[Easy Lemma: An isometry of a line is the identity if it fixes 2 points, a reflection in a point if it fixes one point, and a translation if it fixes no point.]
Hence restricting © to the invariant line gives an isometry of the line which either has a fixed point or does not. If it has two fixed points, then the original © fixed the line, hence was either the identity or a reflection, hence has more than one invariant line, contradiction. If the restriction has one fixed point it is a reflection of the line in that point. Then composing © with reflection of the plane in the line perpendicular to the given line at that point, gives a map R© of the plane that fixes the line, hence is a reflection or the identity. Thus © is either a reflection or a product of two reflections, hence either way does not have just one invariant line, contradiction.Thus the restriction has no fixed points on the invariant line, hence is a translation of that line. Then composing © with a translation back the other way, in the invariant line, fixes the line, hence gives again either the identity, contradiction, or a reflection in that line. The last case is thus the only possible one, whence © itself is a composition of a translation in the invariant line and a reflection in that line, hence a glide along that line. QED.

Proposition 2: A composition of 3 reflections, $R_s R_m R_n$ = ©, in which m is parallel to n, and s is not parallel to m, has exactly one invariant line.
Proof: (By the class at large.) The product $R_m R_n$ is of course a translation T. If s is perpendicular to m, then the product © is a glide by definition and hence has one invariant line. Assume then that s is not perpendicular to m.The invariant line t is obtained from s by translating s backwards by half the translation T. I.e. translate s by (- 1/2) the translation vector of T, to get the line t. Then T translates t up past s exactly as far past s as t was on the other side of s originally. Hence

reflection in s puts t back where it was. Any other line parallel to s is moved by T a distance from s which differs from its original distance, and hence reflection in s puts it back at that new distance from s, hence not where it was. Any line meeting s is translated by T so as to still meet s but at a different point, and hence the reflection of this line still meets s at the new point, hence this reflection does not equal the original line. Consequently there is precisely one invariant line.

Proposition 3: A product $R_sR_mR_n$ = ©, in which m meets n at A, and s does not contain A, has exactly one invariant line.
Proof: This was left as an exercise.

Remark: (This last exercise took me several hours to solve, but my 15 year old son Paul solved it in about 2 minutes.)

Corollary: The product © of any three reflections is either a reflection or a glide; more precisely © is a reflection if the three lines of reflection are all parallel or are all concurrent, and © is a glide otherwise.
Proof: The only thing left to prove after the propositions above are the two cases where the three lines are either all parallel or all concurrent, but these are the easy cases. If they are all concurrent use the fact that the product of the left - most two reflections is a rotation which can be written as a product of form R_kR_n, by a basic theorem on structure of rotations. Then © = $R_kR_nR_n = R_k$. In a similar way the analogous theorem on the structure of translations gives the case where all three lines are parallel. QED.