Investigation of Möbius mappings when they are iterated

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Abstract.

When dealing with *iterated Möbius mappin*gs we need to pay attention to the eigenvalues and eigenvectors of the corresponding matrix. These are important as they show us the properties of the resulting composite Möbius mapping. The eigenvalues in question give two different cases with three and respectively two possibilities per case. In the first case we assume that the eigenvalues are not equal; thus the three different possibilities obtained are from the ratios of the values. If the absolute value of the ratio between the first and the second eigen value is; (i) less than one, the image of z converges to zero, (ii) larger than one, the image of z converges to infinity, (iii) is equal to one, the image rotates in a circle. The second case is when the eigenvalues are equal to each other, which creates two possibilities for the picture of z. The first possibility is represented by an identity, whereas the second possibility shows an image of z at infinity. A summary of our observations is that the composite function, $A^{\infty}(z)$ maps z on a fix point in all cases – either on $B(0)$ or on $B(\infty)$ where B is a conjugate matrix.

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I. Introduction

As we are introduced to a Möbius transformation it is interesting to see how this phenomenon works. We know that Möbius mappings are composite functions that are composed of a rotation and a translation, but what we do not know is *what happens when we repeat a Möbius transformation on a point?* These affine transformations operate in the complex plane and have the form of $z \mapsto \frac{az+b}{cz+d} = A(z)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} a,b,c,d \in \mathbb{C}$.When the mappings occur the image of the point that we are investigating can be of the following four possibilities; (i) a line is mapped onto a line, (ii) a line is mapped onto a circle, (iii) a circle is mapped onto a circle, (iv) a circle is mapped onto a line. Thus, we are going to apply theories that are constructed for real numbers on complex numbers. We can for instance see how the Möbius transformation has its correspondence in matrix form, and we can apply the theory of eigenvalues and their eigenvectors too. These eigenvalues are going to show us what happens to the image of a point after repeated Möbius transformations.

In connection with Möbius transformations it is good to introduce what is meant by the Riemann's sphere. As we are speaking of complex numbers it is hard to avoid the Riemann's sphere, as it is the representative of all the numbers that we are talking about. The sphere can be seen as a one-to-one image of the complex plane, and the north pole of this sphere represents the point of infinity. Thus, with the help of the Riemann's sphere we can feel that infinity is in fact a point.

The aim of this investigation is therefore to see what happens when we iterate a Möbius mapping.

II. Working with Eigenvalues

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \mapsto \dfrac{az+b}{cz+d} = A(z)$, then what happens with $A^n(z)$ if $n \to \infty$? *(0)*

In order to analyze this we need to find the eigenvalues for the following matrix equation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

which we can rewrite into,

$$\begin{pmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

Thus, we are now to find the determinant of the matrix equation. Since we are not looking for the trivial solution of the eigenvectors, the $\det|a-\lambda I|$ must be zero. Therefore the characteristic equation is,

$$\det\begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = 0 \tag{3}$$

and its determinant is,

$$\det\begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = 0 \Leftrightarrow (a-\lambda)(d-\lambda) - bc = 0 \tag{4}$$

We hereby obtain two roots $\lambda_1, \lambda_2 \in \mathbb{C}$.

Consequently, there are therefore two possible cases for the eigenvalues. The first case is when $\lambda_1 \neq \lambda_2$, and the second is when $\lambda_1 = \lambda_2$. We can therefore investigate both cases with chronology, starting with ***case 1***.

III. Case 1. ($\lambda_1 \neq \lambda_2$)

The eigenvalues that we have obtained give restrictions to values of eigenvectors. Thus, the eigenvectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are restricted to: $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \neq \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{C}^2$. We therefore get the relation,

$$A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{5}$$

and

$$A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \tag{6}$$

Hence, *(5)* and *(6)* can be rewritten into,

$$A\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{7}$$

We now want to make $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ the subject, which is displayed in *(8)*,

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{-1} A\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{8}$$

For simplicity we call $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$=*B*. Hence, we can rewrite *(8) into*,

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{9}$$

Thus, the results from *(9)* can be rewritten into,

$$\therefore A = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} \tag{10}$$

We can now test what will happen to *(10)* if we raise it to the power of 2:

$$A^2 = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} \times B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^2 B^{-1} \tag{11}$$

We can also test what will happen to *(11)* if we increase the power by one to get a power of 3.

$$A^3 = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} \times B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} \times B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}B^{-1} = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^3 B^{-1} \tag{12}$$

We can now generalize the results obtained in *(11)* and *(12)* by induction and form our conclusion,

$$\therefore A^n = B\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n B^{-1} \tag{13}$$

(13) can for simplicity be rewritten into,

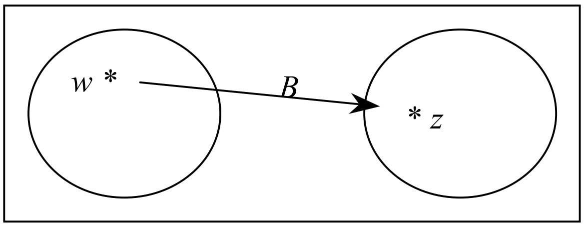
$$A^n = B\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}B^{-1} \tag{14}$$

We have now shown what will happen to the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if we raise it to n.

We can now call the matrix

$$\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} = M \tag{15}$$

Since we know the nature of M's behaviour, we can find out how the conjugate BMB^{-1} behaves.

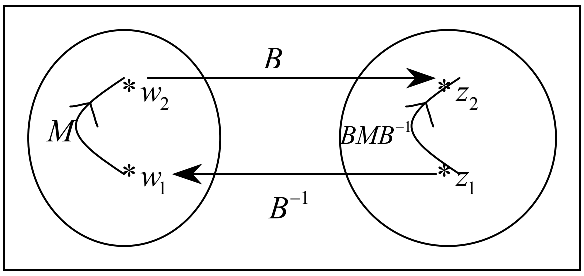


Picture 1

From ***picture* 1** we obtain that $z = B(w)$, and consequently $w = B^{-1}(z)$.

If $M(w) = M\left(B^{-1}(z)\right)$, then $BM(w) = BMB^{-1}(z)$.

We can now draw another picture displaying what happens when M gets involved in the transformation.



Picture 2

We can hereby read from ***picture 2*** the values of z.

$$z_1 = B(w_1) \tag{16}$$

$$z_2 = B(w_2) \tag{17}$$

If we assume that $M(w_1) = w_2$, then we can rewrite the value for z_2 from *(16)*. Thus,

$$z_2 = B(w_2) = B\left(M(w_1)\right) = BM\left(B^{-1}(z)\right) = BMB^{-1}(z_1) \tag{18}$$

A conclusion from this is that BMB^{-1} is the "converter" of M from the w-parameter to the z-parameter.

We may now ask, what happens with $\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$ when $n \to \infty$?

The Möbius transformation in question is,

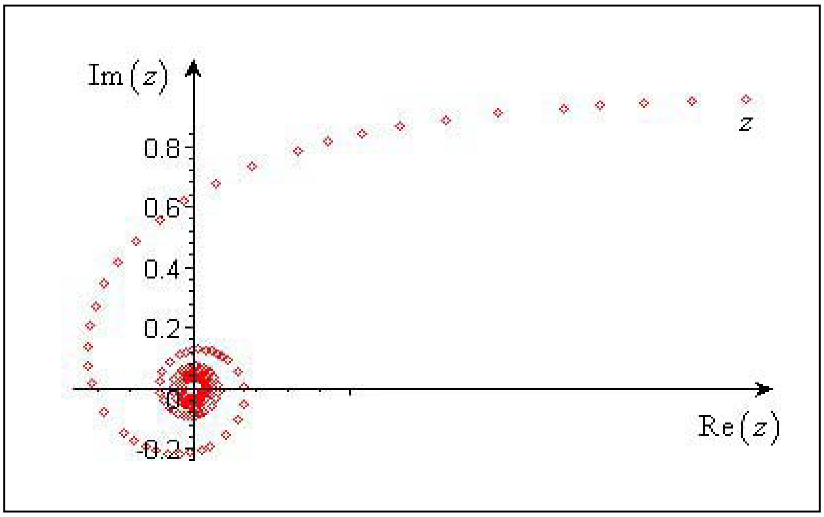
$$z \mapsto \frac{\lambda_1^n z + 0}{0 + \lambda_2^n} = \left(\frac{\lambda_1}{\lambda_2}\right)^n z \qquad (19)$$

Hence there are three possibilities:

(i) $\left|\frac{\lambda_1}{\lambda_2}\right| < 1$

Here $\left(\frac{\lambda_1}{\lambda_2}\right)^n z \to 0$, when $n \to \infty$

Except when $z = \infty$



Picture 3

In ***picture 3*** we can see how the successive image of z converges to 0, as $\left(\frac{\lambda_1}{\lambda_2}\right)^n z \to 0$, $n \to \infty$ in ***(i)***.

The second case **(ii)** is when $\left|\frac{\lambda_1}{\lambda_2}\right| > 1$

Here $\left(\frac{\lambda_1}{\lambda_2}\right)^n z \to \infty$, when $n \to \infty$

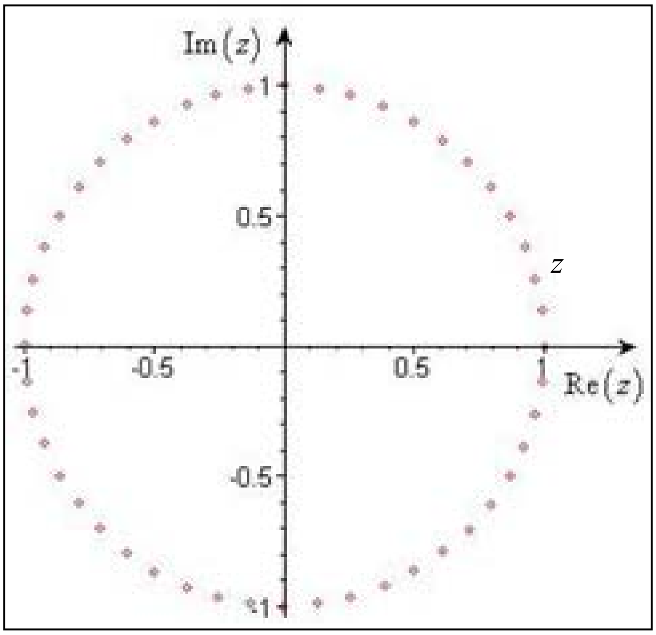
Except when $z = 0$

Once again we can look at ***picture 3*** but think in the opposite manner as for possibility ***(i)***. That is, *z* moves to infinity as $n \to \infty$.

The third option **<u>(iii)</u>** is when $\left|\frac{\lambda_1}{\lambda_2}\right| = 1$

Here we can see all the *z* values rotate in a circle, hence there are finite *z* if $\arg\left(\frac{\lambda_1}{\lambda_2}\right) \in \mathbb{Q}$

Otherwise, there are infinite values of *z*. This is displayed in ***picture 4*** where we can see that there is a finite amount *z* rotating in a circle.



<u>Picture 4</u>

We can now investigate what happens when $\lambda_1 = \lambda_2$, seen in ***case 2***.

<u>IV. Case 2.</u> ($\lambda_1 = \lambda_2$)

Once again we have the Möbius transformation,

$$z \mapsto \frac{az+b}{cz+d} = A(z), \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Now, we assume that the eigenvalues λ_1 and λ_2 are equivalent. Then, we can conjugate using a matrix *K*, where $K = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$ and its inverse, $K^{-1} = \begin{pmatrix} \delta & -\beta \\ -\chi & \alpha \end{pmatrix}$.

We can also assume that the determinant of *K* is:

$\alpha\delta - \beta\chi = 1$, where $\alpha, \beta, \chi, \delta \in \mathbb{C}$

Thus, $K^{-1}AK = \begin{pmatrix} \delta & -\beta \\ -\chi & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$

$$=\begin{pmatrix} * & * \\ -\chi a+\alpha c & -\chi b+\alpha d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$$

$$=\begin{pmatrix} * & * \\ -\chi\alpha a+\alpha^2 c-\chi^2 b+\alpha\chi d & * \end{pmatrix}$$
$$=c\alpha^2+(d-a)\alpha\chi-b\chi^2$$

Hence, we can make $c\alpha^2+(d-a)\alpha\chi-b\chi^2$ equal zero, which can be done by choosing proper values of $K=\begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix}$.

For instance, if we use $\chi=1$, then we need to find a value of α so $c\alpha^2+(d-a)\alpha-b=0$, which works unless, $c=d-a=0$, where $b\neq 0$. (*)

Another alternative can be, $\alpha=1$, where we instead need to find a value of χ so the equation $-b\chi^2+(d-a)\chi+c=0$. (**)

Note however, both (*) and (**) cannot happen, therefore at least one of the choices is possible.

Now, it is easy to find a value of β and δ to make $\alpha\delta-\beta\chi=1$.

We have now made a conjugate of A to form the matrix $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

It enables us now to change notation of the new matrix, which we can call $\begin{pmatrix} q & r \\ 0 & s \end{pmatrix}$. Note however that the eigenvalues of this matrix do not change due to this conjugate, as the characteristic polynomial is,

$$\det\begin{vmatrix} (q-X) & r \\ 0 & (s-X) \end{vmatrix}=(q-X)(s-X)-0=(X-q)(X-s)=(X-\lambda_1)(X-\lambda_1) \quad \textit{(20)}$$

Hence, $q=s=\lambda_1$ gives the matrix, $\begin{pmatrix} \lambda_1 & r \\ 0 & \lambda_1 \end{pmatrix}$

Now, we have two possible possibilities, the first is **(iv)** where we assume that $r=0$. This indicates that the Möbius transformation is

$$z \mapsto \frac{\lambda_1 z + 0}{\lambda_1} = z \tag{21}$$

Hence, it is an Identity transformation, "we understand everything about it".

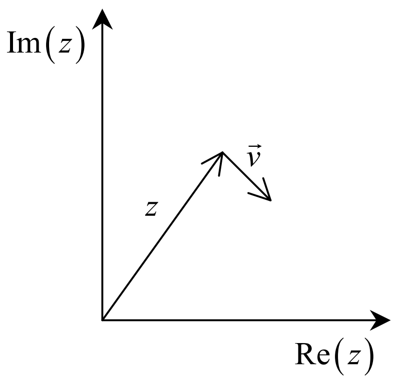
The second possibility **(v)** displays when $r \neq 0$. Here the Möbius transformation is

$$z \mapsto \frac{\lambda_1 z + r}{\lambda_1} = z + \left(\frac{r}{\lambda_1}\right) \tag{22}$$

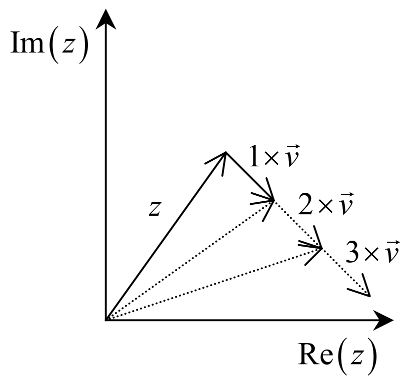
If we now iterate *(22)* *n* times and call $\left(\frac{r}{\lambda_1}\right) = \vec{v}$, we get

$$z \mapsto = z + n\vec{v} \tag{23}$$

We can illustrate this by using argand diagrams and vectors. ***(see picture 5 and 6)***



Picture 5

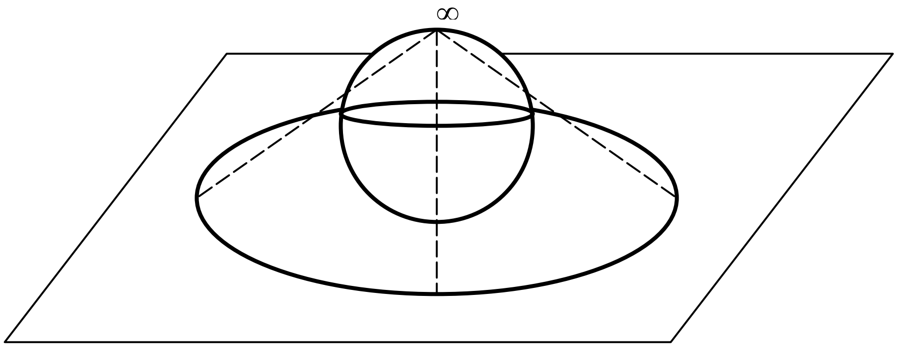


Picture 6

In ***picture 5*** we are given the original values, whereas in ***picture 6*** we see how the image of the original picture is translated $n = 3$ steps. Thus if we are to generalize ***picture 6***, we can see that if $n \to \infty$, then the image of *z* will be at infinity. This image can in fact be seen as a point on the Riemann's sphere, which makes it easier to grasp.

V. Conclusion

As our research is complete we can look at our research question *(0)* and state that the eigenvalues and eigenvectors play a significant role when deciding what will happen when a Möbius transformation is repeated. We gain two possible cases where the first one is when the eigenvalues are different, and the second case when they are equal. ***Case 1*** show that there are three possibilities for the image of *z*, when iterated *n* times as *n* goes towards infinity. If the magnitude of the ratio of the two is larger than one, $\left|\frac{\lambda_1}{\lambda_2}\right| > 1$, then the image of *z* moves in a spiral motion from towards infinity as *n* increases, where *n* is the number of times the Möbius transformation have been iterated. However, if this quotient is less than one, $\left|\frac{\lambda_1}{\lambda_2}\right| < 1$, then the image of *z* moves in a spiral motion towards zero, which is the second possibility. The third possibility is when the ratio is equal to one, $\left|\frac{\lambda_1}{\lambda_2}\right| = 1$, where its consequence is that the image of *z* rotates in the unit circle. In ***Case 2*** we have only two possibilities, as the eigenvalues are equal to each other. The first possibility is when the image of *z* represents an identity, whereas the second possibility maps the image of *z* at infinity, which can be seen as a fix point on the Riemann sphere. This image of infinity is illustrated on the Riemann's sphere, in ***picture*** *7*.



Picture 7

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