Newton-Raphson

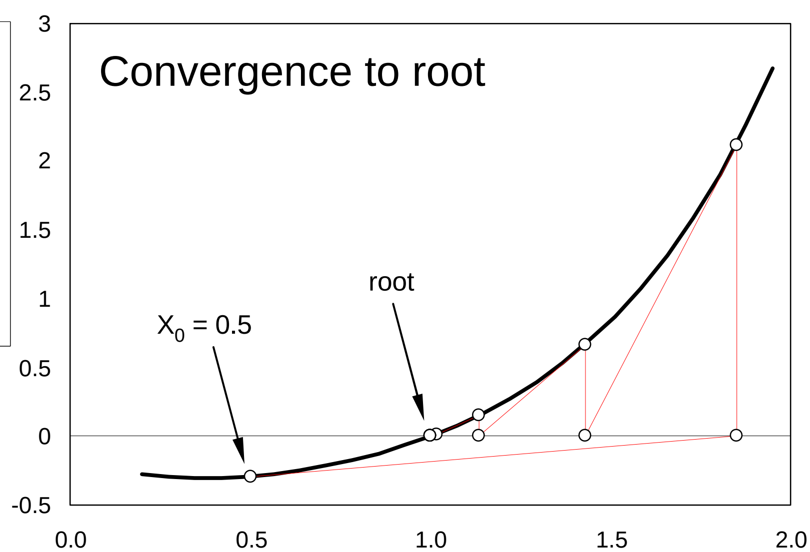
Newton's method, also called the Newton-Raphson method, is a root-finding algorithm for a function ***f(x) = 0*** that uses the first few terms of the Taylor series of the function in the vicinity of a suspected root. Rather than provide the complete derivation, we will provide the final result:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where x_{n+1} is the updated guess for the root of the function based on the previous guess for the root, x_n. Unfortunately, this procedure can be unstable near a horizontal asymptote or inflection point, but if the choice of the initial guess x_0 is sufficiently close to the root, the algorithm can by applied iteratively to converge on the root. Convergence is quadratic.

Example: $f(x) = x^x - 1$ $\quad f'(x) = (x^x)(1+\ln x)$ $\quad x_0 = 0.5$

n	***x n***	***f(x n)***	***f'(x n)***	***x n+1***	***Error***
1	0.50	-0.292893219	0.216977709	1.849876997	8.5E-01
2	1.849876997	2.120192702	5.039482981	1.429160683	4.3E-01
3	1.429160683	0.665849465	2.260703215	1.134628672	1.3E-01
4	1.134628672	0.154087247	1.29985474	1.016086775	1.6E-02
5	1.016086775	0.016347663	1.032567305	1.000254719	2.5E-04
6	1.000254719	0.000254784	1.000509536	1.000000065	6.5E-08
7	1.000000065	6.48653E-08	1.00000013	1	4.2E-15
8	1	4.21885E-15	1	1	0.0E+00
9	1	0	1	1	0.0E+00
10	1	0	1	1	0.0E+00



Newton-Raphson: system of nonlinear equations

The Newton-Raphson method for equations in a single variable has an equivalent for systems of non-linear equations. In matrix notation, this is: $[x_{n+1}] = [x_n] - [J]^{-1}[F_n]$

Where x_{n+1} = updated estimate of the roots of the equations, and J is the Jacobian matrix of partial derivatives. *Example:*

$f_1(x,y) = x^2 + xy - 10 = 0$

$f_2(x,y) = y + 3xy^2 - 57 = 0$

$(x_0, y_0) = (1.5, 3.5)$

$$[J] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x+y & x \\ 3y^2 & 1+6xy \end{bmatrix}$$

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n	$[x_n]$	$[J_n]$	$[J_n]^{-1}$	$[F_n]$	$[J_n]^{-1}[F_n]$	$[x_{n+1}]$
0	$\begin{bmatrix} 1.5 \\ 3.5 \end{bmatrix}$	$\begin{bmatrix} 6.5 & 1.5 \\ 36.75 & 32.5 \end{bmatrix}$	$\begin{bmatrix} 0.2081665 & -0.009608 \\ -0.235388 & 0.0416333 \end{bmatrix}$	$\begin{bmatrix} -2.5 \\ 1.625 \end{bmatrix}$	$\begin{bmatrix} -0.536028823 \\ 0.6561249 \end{bmatrix}$	$\begin{bmatrix} 2.036029 \\ 2.843875 \end{bmatrix}$
1	$\begin{bmatrix} 2.036029 \\ 2.843875 \end{bmatrix}$	$\begin{bmatrix} 6.915933 & 2.036029 \\ 24.26288 & 35.74127 \end{bmatrix}$	$\begin{bmatrix} 0.1807083 & -0.010294 \\ -0.122673 & 0.034967 \end{bmatrix}$	$\begin{bmatrix} -0.064374959 \\ -4.756208497 \end{bmatrix}$	$\begin{bmatrix} 0.037328214 \\ -0.158413463 \end{bmatrix}$	$\begin{bmatrix} 1.998701 \\ 3.002289 \end{bmatrix}$
2	$\begin{bmatrix} 1.998701 \\ 3.002289 \end{bmatrix}$	$\begin{bmatrix} 6.99969 & 1.998701 \\ 27.04121 & 37.00406 \end{bmatrix}$	$\begin{bmatrix} 0.1805343 & -0.009751 \\ -0.131928 & 0.0341499 \end{bmatrix}$	$\begin{bmatrix} -0.004519896 \\ 0.04957115 \end{bmatrix}$	$\begin{bmatrix} -0.001299375 \\ 0.00228915 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2.999999 \end{bmatrix}$
3	$\begin{bmatrix} 2 \\ 2.999999 \end{bmatrix}$	$\begin{bmatrix} 6.999999 & 2 \\ 26.99999 & 36.99999 \end{bmatrix}$	$\begin{bmatrix} 0.1804878 & -0.009756 \\ -0.131707 & 0.0341463 \end{bmatrix}$	$\begin{bmatrix} -1.28609E-06 \\ -2.21399E-05 \end{bmatrix}$	$\begin{bmatrix} -1.61237E-08 \\ -5.86611E-07 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
4	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 7 & 2 \\ 27 & 37 \end{bmatrix}$	$\begin{bmatrix} 0.1804878 & -0.009756 \\ -0.131707 & 0.0341463 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2.23821E-12 \end{bmatrix}$	$\begin{bmatrix} -2.18362E-14 \\ 7.64267E-14 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
5	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 7 & 2 \\ 27 & 37 \end{bmatrix}$	$\begin{bmatrix} 0.1804878 & -0.009756 \\ -0.131707 & 0.0341463 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$