Show that for the Gaussian potential, the cross-section is,

$$V(r)=V_0 e^{-(r/r_0)^2}; \quad \frac{d\sigma}{d\Omega}=\frac{1}{4}\pi r_0{}^2\left(\frac{\mu V_0 r_0{}^2}{\hbar^2}\right)^2 e^{-(qr_0)^2/2} \rightarrow \sigma=\frac{1}{2}\left(\frac{\pi\mu V_0 r_0{}^2}{\hbar^2 k}\right)^2\left(1-e^{-2(kr_0)^2}\right); \qquad \text{[I.1]}$$

Hint: $q^2=2k^2(1-\cos\theta) \leftrightarrow d(\cos\theta)=-\frac{1}{2k^2}d(q^2)=-\frac{q\cdot dq}{k^2}$.

Preliminaries: Amplitude, and consequent cross section, and formula with which to complete the square,

$$f(\Omega)=\frac{-\mu}{2\pi\hbar^2}\int e^{-\mathbf{i}\cdot\mathbf{q}\bullet\mathbf{r}'}V(\mathbf{r}')d^3r'=\frac{-\mu}{2\pi\hbar^2}\int e^{-\mathbf{i}\cdot qr'\cos\theta'}V(r')r'^2dr'd(\cos\theta')d\phi'=\frac{-2\mu}{\hbar^2}\int\frac{\sin qr'}{qr'}V(r')r'^2dr' \qquad \text{[I.2]}$$

Putting in the Gaussian potential to [I.2] and completing the square $\pm\mathbf{i}qr'-(\frac{r'}{r_0})^2=-[\frac{r'\mp\mathbf{i}qr_0{}^2/2}{r_0}]^2$,

$$\frac{f(\Omega)}{-\frac{2\mu V_0}{\hbar^2}}=\int_0^\infty\frac{\sin qr'}{q}e^{-(r'/r_0)^2}r'dr'=\int_0^\infty\frac{e^{\mathbf{i}qr'-(r'/r_0)^2}-e^{-\mathbf{i}qr'-(r'/r_0)^2}}{2\mathbf{i}q}r'dr'=e^{-q^2/(4r_0^2)}\int_0^\infty\frac{\exp(-[\frac{r'-\mathbf{i}qr_0{}^2/2}{r_0}]^2)-\exp(-[\frac{r'+\mathbf{i}qr_0{}^2/2}{r_0}]^2)}{2\mathbf{i}q}r'dr'$$
$$=\frac{1}{4\mathbf{i}q}\left(\int_{\frac{-1}{2}\mathbf{i}qr_0{}^2}^{\infty}e^{-(\rho_-/r_0)^2}(\rho_-+\tfrac{1}{2}\mathbf{i}qr_0{}^2)\cdot d\rho_- -\int_{\frac{+1}{2}\mathbf{i}qr_0{}^2}^{\infty}(\rho_+-\tfrac{1}{2}\mathbf{i}qr_0{}^2)e^{-(\rho_+/r_0)^2}d\rho_+\right) \qquad \text{[I.3]}$$

Something lets you deform a contour…

The following sum of integrals has integrals that are both integrated over straight lines in the complex plane. Deform the contours back to the origin and avoid the singularity at $x\rightarrow\infty$ to prove the integral formula,

$$\int_{-x_0}^{\infty}(x+x_0)e^{-(x/a)^2}dx-\int_{+x_0}^{\infty}(x-x_0)e^{-(x/a)^2}dx=2x_0\int_{-\infty}^{+\infty}e^{-(x/a)^2}dx$$