One way to compare length-measurements in different frames is to compare how a wave-packet looks in different frames.

Let the function f(x), independent of y and z, describe a symmetric profile in the x-coordinate with a definite width w. then, you have the wave-solution,

$$\psi(x,t) = f(x-ct) \qquad \text{[I.1]}$$

as a solution to the wave equation,

$$\Box\psi = 0 \qquad \text{[I.2]}$$

in the unprimed frame.

a) find the function $\psi'(x',t')$ *that describes how this packet looks in the primed frame. The primed frame is moving in the –x direction at velocity v.*

In order to find the function ψ' that is an eigenfunction of $\Box'$, we only have [I.2] as a true statement. Thus, we need to Lorentz-transform the wave equation [I.2] and see if terms of $\Box\psi$ pop up.

the Lorentz-transforms (and their inverses) in the x-direction are,

$$t' = \gamma(t - \tfrac{1}{c^2}vx) \quad \leftrightarrow \quad t = \gamma(t' + \tfrac{1}{c^2}vx') \qquad \text{[I.3]}$$

$$x' = \gamma(x - vt) \quad \leftrightarrow \quad x = \gamma(x' + vt') \qquad \text{[I.4]}$$

The $\Box$ operator of [I.2] transforms from [I.3] and [I.4] as,

$$\Box \to \Box' = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \nabla'^2 \qquad \text{[I.5]}$$

how do partial derivatives transform? They should transform linearly with the jacobian-matrix as the mapping device,

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x'} \end{bmatrix} = \begin{bmatrix} \frac{\partial t}{\partial t'} & \frac{\partial x}{\partial t'} \\ \frac{\partial t}{\partial x'} & \frac{\partial x}{\partial x'} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{bmatrix} \qquad \text{[I.6]}$$

With [I.3] and [I.4], this jacobian is computed as,

$$\begin{bmatrix} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x'} \end{bmatrix} = \gamma \begin{bmatrix} 1 & v \\ \frac{1}{c^2}v & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{bmatrix} = \gamma \begin{bmatrix} \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \\ \frac{v}{c^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \end{bmatrix} \qquad \text{[I.7]}$$

Putting [I.7] into [I.5], we get,

$$\Box \to \Box' = \frac{1}{c^2}\gamma^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)^2 - \gamma^2\left(\frac{v}{c^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \qquad \text{[I.8]}$$

A cross term disappears from [I.8] when we FOIL out the squares (from $(\frac{1}{c}\frac{\partial}{\partial t'})^2$ and $(\frac{\partial}{\partial x'})^2$; first two terms),

$$\Box' = \frac{\gamma^2}{c^2}\left(\frac{\partial^2}{\partial t^2} + v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x \partial t}\right) - \gamma^2\left(\frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + 2\frac{v}{c^2}\frac{\partial^2}{\partial x \partial t}\right) - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \gamma^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{v^2}{c^2}\frac{\partial^2}{\partial x^2} + 2\frac{v}{c^2}\frac{\partial^2}{\partial x \partial t} - \left(\frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + 2\frac{v}{c^2}\frac{\partial^2}{\partial x \partial t}\right)\right) - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \qquad [I.9]$$

$$\Box' = \gamma^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{v^2}{c^2}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\right) - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \gamma^2\left(\frac{1}{c^2} - \frac{v^2}{c^4}\right)\frac{\partial^2}{\partial t^2} + \gamma^2\left(\frac{v^2}{c^2} - 1\right)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

The quantities $\gamma^2(\frac{1}{c^2} - \frac{v^2}{c^4})$ and $\gamma^2(\frac{v^2}{c^2} - 1)$ then pleasantly collapse, and we demonstrate invariance of [I.2] under Lorentz transform,

$$\Box' = \frac{\frac{1}{c^2}}{1-(\frac{v}{c})^2}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial t^2} + \frac{-1}{1-(\frac{v}{c})^2}\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \boxed{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \Box} \qquad [I.10]$$

Because $\Box = \Box'$, we can write [I.2] as,

$$\Box\psi = \Box'\psi \qquad [I.11]$$

show that $\psi'(x', t')$ solves the wave equation in the primed system.

b) what is the width w' of this profile as determined in the primed system? Notice that w'/w is <u>not</u> the usual Lorentz-contraction factor…

c) to understand this difference: consider a rod of rest-length L_0 parallel to the x-axis, moving in the +x direction with velocity u in the unprimed frame. What is its velocity in the primed frame?

d) calculate the lengths L, L' of the rod as seen in the primed and unprimed systems respectively.

Compare w'/w and L'/L, and comment…