Dimensions in Special Relativity Theory - *a Euclidean Interpretation**

R.F.J. van Linden
Smeetsstraat 56, 6171 VD Stein, NETHERLANDS
e-mail r.vanlinden@freeler.nl, web http://www.euclideanrelativity.com

September 2005

Abstract

A Euclidean interpretation of special relativity is given wherein proper time τ acts as the fourth Euclidean coordinate, and time t becomes a fifth Euclidean dimension. Velocity components in both space and time are formalized while their vector sum in four dimensions has invariant magnitude c. Classical equations are derived from this Euclidean concept. The velocity addition formula shows a deviation from the standard one; an analysis and justification is given for that.

 PACS 03.30.+p.

1 Introduction

Euclidean relativity, both special and general, is steadily gaining attention as a viable alternative to the Minkowski framework, after the works of a number of authors. Amongst others Montanus [1,2], Gersten [3] and Almeida [4], have paved the way. Its history goes further back, as early as 1963 when Robert d'E Atkinson [5] first proposed Euclidean general relativity.

The version in the present paper emphasizes extending the notion of velocity to the time dimension. Next, the consistency of this concept in 4D Euclidean space is shown with the classical Lorentz transformations, after which the major *inconsistency* with classical special relativity, the velocity addition formula, is addressed. Following paragraphs treat energy and momentum in 4D Euclidean space, partly using methods of relativistic Lagrangian formalism already explored by others after which some Euclidean 4-vectors are established.

A simplified and popularized version is available that will get you in the 'right mood'. It can be found on the web at http://www.euclideanrelativity.com.

2 The Time Dimension

Minkowski interpretations of special relativity treat time differently from spatial dimensions, showing from the Minkowski metric where the time component is given the opposite sign. Some alternative interpretations (*e.g.* [1-4]) seek positive definite Euclidean metrics for space-time. Also in this article, the time dimension will be treated as a regular fourth dimension in Euclidean space-time.

If time is considered a fourth spatial dimension, then it must show properties similar to those found in the other three. In there we encounter properties like length, speed, acceleration, curvature *etc.*, expressed respectively as s, ds/dt, d^2s/dt^2, R^a_{bcd} *etc.* Of those properties, a single one can be measured relatively easily in the time dimension: the 'length' or timeduration Δt. That raises the question of how a hypothetical speed in time, let us call it χ, should be expressed mathematically. In [6], Greene has given a derivation of an expression that can be used as the velocity component in the Euclidean time dimension. Rewriting the usual Minkowski invariant

$$c^2 = (dct/d\tau)^2 - (dx/d\tau)^2 - (dy/d\tau)^2 - (dz/d\tau)^2 \tag{1}$$

into Euclidean form:

$$c^2 = (cd\tau/dt)^2 + (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 \tag{2}$$

one arrives at the temporal velocity component

$$\chi = cd\tau/dt \tag{3}$$

This clearly defines τ as the coordinate for the fourth Euclidean dimension, and it says that the velocity components in all *four* dimensions involve derivatives with respect to t, which then can no longer represent the fourth dimension. It can only be an extra, fifth dimension, x_5 (provided we index the other four x_1, x_2, x_3, and x_4 respectively, with $\tau = x_4$). This fifth dimension is sometimes treated as a parameter in Euclidean approaches similar to special relativity, *e.g.* in [1,2], but here it will be treated as a genuine extra Euclidean dimension. A general expression for speed in the time dimension (henceforth refereed to as time-speed) is now:

$$\chi = cdx_4/dx_5 \tag{4}$$

while the scalar value of time-speed χ is

$$\chi = \sqrt{c^2 - v^2} \tag{5}$$

The general expression for spatial velocity components in 4D Euclidean space-time is

$$v_i = dx_i/dx_5 \tag{6}$$

3 Using Time-Speed in Special Relativity

It will be shown that the Lorentz transformation equations for length and time can be reproduced from the Euclidean context.

Maintaining orthogonality for all Euclidean dimensions, Eqs. (2) and (5) imply that the axes for the proper time dimension and the spatial dimension in the direction of the initial motion must have *rotated* for the moving object, as seen from the rest frame of the observer, in fact defining Lorentz transformations as rotations in SO(4). See also [1], where this is referred to as a Relative Euclidean Space-Time. In the approach that follows now, these axes will therefor (unlike in the Minkowski diagram) both rotate in the same direction, clockwise or counter clockwise, depending on the direction of the motion. The diagrams in Fig. 1 and Fig. 2 should illustrate this.

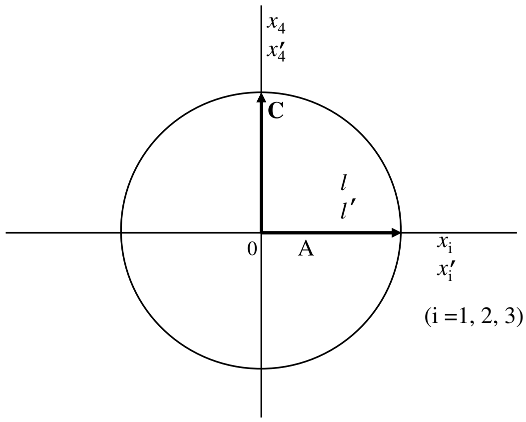

Figure 1: 4D representation of an observer at O and an object A, both at rest.

Figure 1 depicts an object A at rest together with an observer at O, also at rest. The horizontal axis shows both the spatial dimensions x'_i, $i = 1, 2, 3$, for the object A as well as the spatial dimensions x_i for the observer. The vertical axis shows both time dimensions with notation conform Eq. (2), so $x_4 = c\tau$. Due to object A being at rest, relative to the observer, the axes overlap. The circle is just a tool to better show the rotation that will be depicted in Fig. 2.

Definitions are as follows:

- Vector $\mathbf{C}$ indicates the 4D velocity, having magnitude c, of object A.
- Vector $\mathbf{V}$, of magnitude v, and $\mathbf{X}$, of magnitude χ, are the projections of this velocity $\mathbf{C}$ on, respectively, the spatial dimensions and the proper time dimension of the observer.
- l' indicates the proper length of object A in the spatial direction x'_i in the rest frame of object A (in this example l' is also set to c).

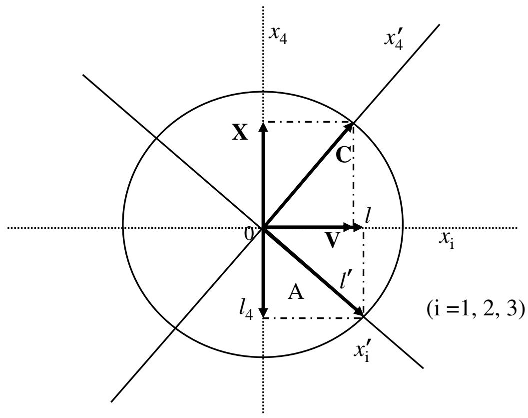

Figure 2: Object A in motion relative to observer. The dimensional axes of object A have rotated relative to the observer.

- l and l_4 are, respectively, the projections of this proper length on the spatial dimensions and the proper time dimension of the observer.

In Fig. 2, object A moves with speed v relative to the observer. This leads to a relative rotation of dimensions x'_4 and x'_i such that $\mathbf{V}$ is the projection of the original 4D velocity $\mathbf{C}$ of object A on the x_i axis of the observer at rest. The situation is examined at the instant where $x_i = x'_i = x_4 = x'_4 = 0$.

The Lorentz transformation equation for x is

$$x' = \gamma(x - vt) \tag{7}$$

where

$$\gamma = 1/\sqrt{1 - v^2/c^2} \tag{8}$$

but this factor can also be written as

$$\gamma = c/\sqrt{c^2 - v^2} = c/\chi \tag{9}$$

leading to

$$x' = c(x - vt)/\chi \tag{10}$$

At $t = 0$, the length of object A will be contracted, as measured by the observer, according to

$$x = x'\chi/c \tag{11}$$

so the contraction of length l can be written as

$$l = l'\chi/c \tag{12}$$

which shows that l, as measured by the observer at rest, is indeed the goniometric projection of the proper length l' on the x_i axis.

Arrow l_4 is the projected 'length' component of the moving object A on the proper time axis x_4 of the observer as a result of the rotation of the dimension x'_i. This length is the manifestation of the difference in proper time (the non-simultaneity) between the endpoints of object A in motion according to the Lorentz transformation equation for time:

$$t' = \gamma(t - vx/c^2) \tag{13}$$

and can be interpreted as a rotation 'out of space' of the proper length l' towards the negative axis of x_4. At $t = 0$ the proper-time difference between tail and head of arrow l will be

$$t' = -\gamma vl/c^2 = -lv/c\chi \tag{14}$$

From $l = l'\chi/c$ and $l_4 = l'v/c$ it follows that

$$l_4 = -ct' \tag{15}$$

which confirms that l_4 represents the proper-time difference in object A. The factor c results from the choice of units for space and time.

Summarizing, from the perspective of the observer, the proper length l' of object A is decomposed in the components l and l_4 according to:

$$l'^2 = l^2 + l_4^2 \tag{16}$$

and so is also the 4D speed c of the object decomposed in the components χ and v:

$$c^2 = \chi^2 + v^2. \tag{17}$$

Equation (16) thus combines Eqs. (7) and (13) into a single Pythagorean equation in four dimensions.

4 Relativistic Addition of Velocities

It appears that the Euclidean approach as used in the previous Section does not yield the same equation for relativistic addition of velocities as used in special relativity. Although this particular point may be a serious obstacle to the acceptation of this proposal, it obviously is necessary to point it out.

Figure 3 depicts a situation with three reference frames: a stationary unprimed frame x, a moving primed frame x' and a third, double primed frame x'' of an object that moves relative to both other frames, x and x'. Each frame has dimensional axes rotated relative to the other frames as a result of the relative motion.

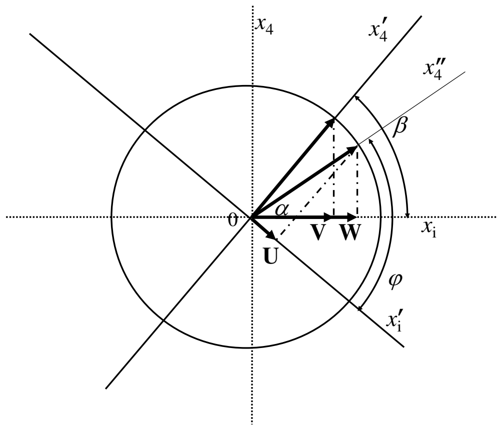

Figure 3: Relativistic addition of velocities in three reference frames, each with rotated dimensional axes relative to each other.

- Vector **V** of magnitude v is the spatial velocity of an observer with rest frame x' as measured by an observer with rest frame x.
- Vector **W** of magnitude w is the spatial velocity of a third object as measured by the observer with rest frame x.
- Vector **U** of magnitude u is the spatial velocity of that same object but now as measured by the observer with rest frame x'.

When u, v, and w are parallel, the classical relation between them is:

$$w = \frac{u + v}{1 + uv/c^2} \tag{18}$$

If we apply the approach as used consistently until now it yields the expression:

$$\begin{aligned} w &= c\cos(-\alpha) = c\sin(\frac{1}{2}\pi + \alpha) \\ &= c\sin(\beta + \varphi) = c(\cos\varphi\sin\beta + \cos\beta\sin\varphi) \\ &= u\sqrt{1 - v^2/c^2} + v\sqrt{1 - u^2/c^2} \end{aligned} \tag{19}$$

This expression is not nearly similar to the classical expression in Eq. (18).

Like Eq. (18), Eq. (19) still limits the speeds as measured by both observers to the maximum of c, which is also clear by inspection of the Figure. Some remarks will be made now on the probability of either of the equations to be the right one:

1. Equation (18) is in fact based on the universality of light speed and the basis for reasoning is that an object, *e.g.* a photon, having speed c for an observer in frame x will still have that same speed c for an observer in frame x'. This is one of Einstein's original postulates and also in this Euclidean approach it will still be maintained as a valid postulate, which essentially means that the photons velocity vector, as measured from the moving frame, must have rotated *along* with that frame. The third object, having speed w, as measured from frame x, is not a photon but a mass-carrying particle for which such a rotation apparently does not apply. It must therefor be emphasized that Eq. (19) for now may only be applied to mass-carrying particles.

2. Equation (18) shows a discontinuity that is unusual in physics. In Fig. 4, Eq. (18) is plotted for the situation where u always equals v.

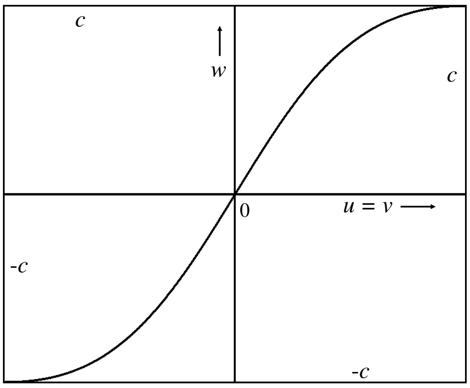

Figure 4: Graph of classical equation for relativistic addition of velocities.

With u and v nearing c, the resulting w will also near c, which is in accordance with the classical view. But if (as a matter of mathematical experiment) the range of u and v is extended beyond the maximum value of c then the plot looks like depicted in Fig. 5.

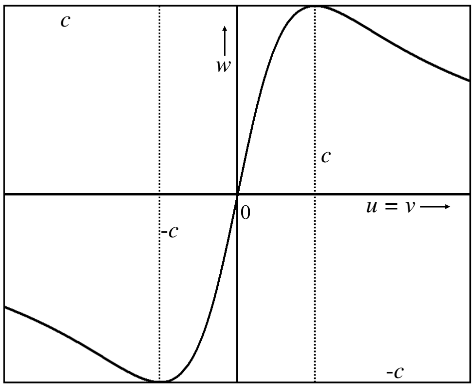

Figure 5: Classical graph for relativistic addition of velocities with hypothetical (superluminal) extensions.

The part from Fig. 4 can still be recognized but it is clear now that this actually forms part of a continuous function that extends beyond c. The part beyond $u = v = c$ may not be used, solely because the classical function is not defined, nor ever shown to be valid, for such superluminal extensions (actually the space-like quadrants in the classical light cone). This fact strongly suggests that the graph from Fig. 4 is an approximation of the real function.

Finally, both Eqs. (18) and (19) are plotted together in Fig. 6.

Equation (19) is almost identical for speeds below about $c/2$ but begins to deviate at higher speeds. The top of Eq. (19) corresponds to $u = v = c/\sqrt{2}$. From the circle diagram in Fig. 3 it shows that the time-speed of the object, as measured from frame x, then becomes zero. Equation (19) further shows *decreasing* values for w in situations where the values of u and v go beyond $c/\sqrt{2}$ (the frame of the moving object then rotates beyond $\pi/2$ relative to frame x). It turns out that in that case the corresponding time-speed for the object becomes negative. (This situation might be related to