

We shall now prove the existence of  $n$ th roots of positive reals. This proof will show how the difficulty pointed out in the Introduction (irrationality of  $\sqrt{2}$ ) can be handled in  $R$ .

**1.21 Theorem** For every real  $x > 0$  and every integer  $n > 0$  there is one and only one positive real  $y$  such that  $y^n = x$ .

This number  $y$  is written  $\sqrt[n]{x}$  or  $x^{1/n}$ .

**Proof** That there is at most one such  $y$  is clear, since  $0 < y_1 < y_2$  implies  $y_1^n < y_2^n$ .

Let  $E$  be the set consisting of all positive real numbers  $t$  such that  $t^n < x$ .

If  $t = x/(1+x)$  then  $0 \leq t < 1$ . Hence  $t^n \leq t < x$ . Thus  $t \in E$ , and  $E$  is not empty.

If  $t > 1+x$  then  $t^n \geq t > x$ , so that  $t \notin E$ . Thus  $1+x$  is an upper bound of  $E$ .

Hence Theorem 1.19 implies the existence of

$$y = \sup E.$$

To prove that  $y^n = x$  we will show that each of the inequalities  $y^n < x$  and  $y^n > x$  leads to a contradiction.

The identity  $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$  yields the inequality

$$b^n - a^n < (b-a)nb^{n-1}$$

when  $0 < a < b$ .

Assume  $y^n < x$ . Choose  $h$  so that  $0 < h < 1$  and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put  $a = y$ ,  $b = y + h$ . Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus  $(y+h)^n < x$ , and  $y+h \in E$ . Since  $y+h > y$ , this contradicts the fact that  $y$  is an upper bound of  $E$ .

Assume  $y^n > x$ . Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then  $0 < k < y$ . If  $t \geq y - k$ , we conclude that

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

Thus  $t^n > x$ , and  $t \notin E$ . It follows that  $y - k$  is an upper bound of  $E$ .

But  $y - k < y$ , which contradicts the fact that  $y$  is the least upper bound of  $E$ .

Hence  $y^n = x$ , and the proof is complete.

**Corollary** If  $a$  and  $b$  are positive real numbers and  $n$  is a positive integer, then

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

**Proof** Put  $\alpha = a^{1/n}$ ,  $\beta = b^{1/n}$ . Then

$$ab = \alpha^n \beta^n = (\alpha\beta)^n,$$

since multiplication is commutative. [Axiom (M2) in Definition 1.12.] The uniqueness assertion of Theorem 1.21 shows therefore that

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}.$$

**1.22 Decimals** We conclude this section by pointing out the relation between real numbers and decimals.

Let  $x > 0$  be real. Let  $n_0$  be the largest integer such that  $n_0 \leq x$ . (Note that the existence of  $n_0$  depends on the archimedean property of  $R$ .) Having chosen  $n_0, n_1, \dots, n_{k-1}$ , let  $n_k$  be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Let  $E$  be the set of these numbers

$$(5) \quad n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \quad (k = 0, 1, 2, \dots).$$

Then  $x = \sup E$ . The decimal expansion of  $x$  is

$$(6) \quad n_0 \cdot n_1 n_2 n_3 \cdots$$

Conversely, for any infinite decimal (6) the set  $E$  of numbers (5) is bounded above, and (6) is the decimal expansion of  $\sup E$ .

Since we shall never use decimals, we do not enter into a detailed discussion.

## THE EXTENDED REAL NUMBER SYSTEM

**1.23 Definition** The extended real number system consists of the real field  $R$  and two symbols,  $+\infty$  and  $-\infty$ . We preserve the original order in  $R$ , and define

$$-\infty < x < +\infty$$

for every  $x \in R$ .